Journal of Pure and Applied Algebra 8 (1976) 289-336

MORE ON THE ARCHIMEDEAN RELATION ON THE FREE SEMIGROUP

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Communicated by J. Rhodes
Received 17 October 1975

0. Introduction

Let S be a semigroup and $a,b \in S$. We define a—b if we can solve $xay = b^i$ and $sbt = a^j$ for some $x,y,s,t \in S^1$ and some positive integers i and j. '—' is called the archimedean relation on S. The archimedean relation arises naturally in the theory of abstract semigroups [13,1,14,7,12,8]. In this paper we continue from [9] the detailed study of the archimedean relation and the corresponding graph on the free semigroup $\mathcal{F}$. In Section 2, we generalize some results of [9] on irreducible sequences and polygons. Since the graph ($\mathcal{F}$, —) contains the countably infinite complete graph as an induced subgraph, it is trivial that a connected graph is a subgraph of ($\mathcal{F}$, —) if and only if it is at most countable. So induced subgraphs of ($\mathcal{F}$, —) are the correct, non-trivial things to study. In [9] we showed that any polygon is an induced subgraph of any free semigroup on more than one letter. Based on this, John Rhodes posed to the author the question of whether every finite graph is an induced subgraph of some free semigroup. In Section 4, we give an example of a finite, astronomically large graph which is not an induced subgraph of any free semigroup. The proof, at least at this stage of development of the theory, is very difficult. In Section 3, we see that as far as studying induced subgraphs goes, the number of letters of the free semigroup makes no difference as long as it is finite and greater than 1. Of course,
semigroup makes no difference as long as it is finite and greater than 1. Of course,
for other questions the number of letters does matter. In Section 5 we study the archimedean relation via equations in the free semigroup. The very complicated nature of the archimedean relation then becomes transparent. In Section 6 we study generalizations of some of our results to free product of copies of positive reals or rationals. In Section 7 we use the results of Section 6 to prove that any finite graph which is obtained by putting together polygons, trees and complete graphs in certain

* Research partially supported by NSF Grant GP-37492X.

trivial ways is an induced subgraph of any free semigroup on more than one letter.

1. Preliminaries

Throughout this paper $\mathbf{Z}, \mathbf{N}, \mathbf{Z}^+, \mathbf{Q}, \mathbf{Q}^+, \mathbf{R}$ and $\mathbf{R}^+$ will denote the sets of integers, non-negative integers, positive integers, rationals, positive rationals, reals and positive reals respectively. If X is a set, $|X|$ will denote the cardinality of X. If S is a semigroup then $S^1 = S \cup \{1\}$ with obvious multiplication. If $a \in S$, then $\langle\langle a \rangle\rangle = \{a^i \mid i \in \mathbf{Z}^+\}$. On the other hand we use $\langle\ \rangle$ to describe sequences in S. Throughout this paper $\mathcal{F} = \mathcal{F}(X)$ will denote the free semigroup (without identity) on some non-empty set X. If $X = \{A_1, ..., A_n\}$ is finite we sometimes write $\mathcal{F}(A_1, ..., A_n)$ instead of $\mathcal{F}(X)$ and we let $\mathcal{C} = \mathcal{C}(X) = \mathcal{C}(A_1, ..., A_n)$ be the *free content* on X which is the subsemigroup of $\mathcal{F}$ consisting of all words which involve each A_i at least once. $\mathcal{C}(X)$ is an ideal of $\mathcal{F}(X)$. If $\omega \in \mathcal{F}$ then $|\omega|$ denotes the length of ω. In $\mathcal{F}^1$ we set $|1| = 0$. If $\omega \in \mathcal{F}^1$ then ω^0 denotes 1. Also for $\omega \in \mathcal{F}$ the *content* of ω is the unique $\mathcal{C}(Y)$, $Y \subseteq X$ such that $\omega \in \mathcal{C}(Y)$. Evidently Y is just the set of letters appearing in ω.

Definition. Let S be a semigroup and $a, b \in S$.

(1) $a | b$ if $b = xay$ for some $x, y \in S^1$.

(2) $a |_{\mathrm{i}} b$ if $b = ay$ for some $y \in S^1$.

(3) $a |_{\mathrm{f}} b$ if $b = xa$ for some $x \in S^1$.

(4) $a \to b$ if $a | b^i$ for some $i \in \mathbf{Z}^+$.

(5) $a — b$ if $a \to b \to a$.

(6) A finite sequence $\langle x_i \rangle_{i=1}^n$ is said to be a sequence between a and b if $a — x_1 — ... — x_n — b$. By $n = 0$ or $\langle x_i \rangle_{i=1}^n$ empty we mean $a — b$; $\langle x_i \rangle_{i=1}^n$ is irreducible [1)] if $n > 0$ and no proper subsequence of $\langle x_i \rangle_{i=1}^n$ (including the empty sequence) is a sequence between a and b. $\langle x_i \rangle_{i=1}^n$ is minimal if $n > 0$ and no sequence of smaller length (including length 0) exists between a and b.

We call '—' the *archimedean relation* on S and $(S, —)$ the *archimedean graph* of S. The archimedean relation is symmetric and reflexive but in general is not transitive. The relations $|$, $|_{\mathrm{i}}$ and $|_{\mathrm{f}}$ are reflexive and transitive; $|_{\mathrm{i}} \subseteq |$ and $|_{\mathrm{f}} \subseteq |$. Also $| \subseteq \to$; $— \subseteq \to$. In case of $\mathcal{F}^1$ the relations $|$, $|_{\mathrm{i}}$ and $|_{\mathrm{f}}$ are partial orders and have the meaning of being segment of, initial segment of and final segment of respectively. Also if $u, v \in \mathcal{F}^1$ then $u | v$ implies $|u| \leqslant |v|$ and $u | v$, $|u| = |v|$ together imply $u = v$. By [11,8] the connected components of the graph $(\mathcal{F}(X), —)$, where X is a non-empty set, are $\mathcal{C}(W)$ as W ranges through all finite non-empty subsets of X. If $Y \subseteq X$, $Y \neq \emptyset$, $\omega_1, \omega_2 \in \mathcal{F}(Y)$ then $\omega_1 | \omega_2$ (respectively $\omega_1 |_{\mathrm{i}} \omega_2$, $\omega_1 |_{\mathrm{f}} \omega_2$, $\omega_1 \to \omega_2$, $\omega_1 — \omega_2$) in $\mathcal{F}(X)^1$ if and only if $\omega_1 | \omega_2$ (respectively $\omega_1 |_{\mathrm{i}} \omega_2$, $\omega_1 |_{\mathrm{f}} \omega_2$, $\omega_1 \to \omega_2$, $\omega_1 — \omega_2$) in $\mathcal{F}(Y)$. On the other hand if Y is a non-empty finite subset

1) in [9] we used the term 'indecomposable' instead.

of X and $\omega_1, \omega_2 \in \mathcal{C}(Y)$ then it can happen that $\omega_1 | \omega_2$ in $\mathcal{F}(Y)$ but $\omega_1 \nmid \omega_2$ in $\mathcal{C}(Y)$. However, $\omega_1 | \omega_2$ in $\mathcal{F}(Y)$ implies $\omega_1 | \omega_2^3$ in $\mathcal{C}(Y)$. Thus $\omega_1 \to \omega_2$ in $\mathcal{F}(Y)$ if and only if $\omega_1 \to \omega_2$ in $\mathcal{C}(Y)$ and $\omega_1 — \omega_2$ in $\mathcal{F}(Y)$ if and only if $\omega_1 — \omega_2$ in $\mathcal{C}(Y)$. Thus there is no ambiguity when dealing with '$\to$' or '—' in $\mathcal{C}(Y)$. Throughout this paper (unless otherwise stated) when we talk about the relations $|$, $|_i$, $|_f$ for elements in $\mathcal{C}(Y)$ we are talking about these relations in the bigger semigroup $\mathcal{F}(Y)$ (equivalently $\mathcal{F}(X)$, $Y \subseteq X$), and *not* in $\mathcal{C}(Y)$.

Following [6] $\omega \in \mathcal{F}$ is said to be *primitive* if $\omega \in \langle\langle U \rangle\rangle$, $U \in \mathcal{F}$ implies $\omega = U$.

Lemma 1.1 [4,5,6] (i). *Let $A,B,C,D \in \mathcal{F}^1$ such that $AB = CD$. Then $|A| \leqslant |C|$ iff $|D| \leqslant |B|$ iff $A \mid_i C$ iff $D \mid_f B$. $|A| = |C|$ iff $|D| = |B|$ iff $A = C$ iff $B = D$.*

(ii) *If $\omega \in \mathcal{F}$ then $\omega \in \langle\langle U \rangle\rangle$ for some primitive $U \in \mathcal{F}$. Moreover U is unique and is called the primitive root of ω.*

(iii) *If $A,B \in \mathcal{F}$ and $AB = BA$ then $A,B \in \langle\langle U \rangle\rangle$ for some primitive $U \in \mathcal{F}$.*

(iv) *If $U \in \mathcal{F}$, $A \in \mathcal{F}^1$, U primitive and $AU = UA$, then $A \in \langle\langle U \rangle\rangle^1$.*

(v) *If $A,B \in \mathcal{F}^1$, $U \in \mathcal{F}$, U primitive, then $AB = U = BA$ implies $A = 1$ or $B = 1$.*

(vi) *Let $A, C \in \mathcal{F}$, $B \in \mathcal{F}^1$, $AB = BC$. Then there exist $a,b \in \mathcal{F}^1$ such that $A = ab$, $C = ba$ and $B = (ab)^i a$ for some $i \in \mathbf{N}$.*

Lemma 1.2. (i) *Let $A,B,C,D,y \in \mathcal{F}^1$. Then $AB \mid_i AC$ iff $B \mid_i C$; $BA \mid_f CA$ iff $B \mid_f C$. If $A \mid_i BC$ and $|B| \leqslant |A|$, then $B \mid_i A$. If $A \mid_f C$ and $B \mid_i D$, then $AyB | CyD$.*

(ii) *Let $A,B \in \mathcal{F}^1$, $U \in \mathcal{F}$, $k \in \mathbf{N}$ such that $A \mid_i BU^k$, $|B| \leqslant |A|$. Then there exist $D,F \in \mathcal{F}^1$, $r \in \mathbf{N}$ such that $r \leqslant k$, $A = BU^rD$, $U = DF$ and $|D| < |U|$.*

(iii) *Let $A,B \in \mathcal{F}^1$, $U \in \mathcal{F}$ such that $AB \in \langle\langle U \rangle\rangle^1$. Then $A \in \langle\langle U \rangle\rangle^1$ iff $B \in \langle\langle U \rangle\rangle^1$.*

(iv) *Let $A,B \in \mathcal{F}^1$, $U \in \mathcal{F}$, $r \in \mathbf{Z}^+$, U primitive. Then $AU^rB \in \langle\langle U \rangle\rangle$ implies that $A,B \in \langle\langle U \rangle\rangle^1$.*

(v) *Suppose $A_1, A_2, B_1, B_2, \omega \in \mathcal{F}^1$, $A_1 \neq A_2$, $A_1 \omega B_1 = A_2 \omega B_2$. Then there exists $a,b \in \mathcal{F}^1$, $j \in \mathbf{N}$ such that $\omega = (ab)^j a$, $|ab| = \big|\, |A_1| - |A_2| \,\big|$.*

(vi) *If $\omega \in \mathcal{F}$ and u the primitive root of ω, then the content of ω = the content of u.*

(vii) *Suppose $U \in \mathcal{F}$, $x \in \mathcal{F}^1$ and U is primitive. Then $UxU \to U$ implies $x \in \langle\langle U \rangle\rangle^1$.*

Proof (i). The first statement follows from cancellation in $\mathcal{F}$. If $A \mid_i BC$ and $|B| \leqslant |A|$, then $AL = BC$ for some $L \in \mathcal{F}^1$ whence $B \mid_i A$. If $A \mid_f C$ and $B \mid_i D$, then for some $L_1, L_2 \in \mathcal{F}^1$, $L_1 A = C$ and $BL_2 = D$. So $L_1 AyBL_2 = CyD$ and $AyB | CyD$.

(ii) By (i), $B \mid_i A \mid_i BU^k$. So for some $A_1, C \in \mathcal{F}^1$, $A = BA_1$ and $AC = BU^k$. Thus $A_1 C = U^k$ and $|A_1| \leqslant |U^k|$. Since $U \neq 1$, $|A_1| < |U^{k+1}|$. So for some $r \in \mathbf{N}$, $r \leqslant k$, $|U^r| \leqslant |A_1| < |U^{r+1}|$. So for some $D \in \mathcal{F}^1$, $A_1 = U^rD$ and $|D| < |U|$. So $DC = U^{k-r}$. If $k - r = 0$, then $D = 1$ and otherwise $D \mid_i U$. So in any case $D \mid_i U$ and $A = BA_1 = BU^rD$.

(iii) $AB = U^r$ for some $r \in \mathbf{N}$. $A = U^s$, $s \in \mathbf{N}$ implies (since $U \neq 1$) that $s \leqslant r$ and $B = U^{r-s} \in \langle\langle U \rangle\rangle^1$. The other direction is similarly proved.

(iv) There exists $s \in \mathbf{Z}^+$ such that $AU^rB = U^s$. By (ii) there exist $D, F \in \mathcal{F}^1$, $t \in \mathbf{N}$ such that $A = U^tD$, $DF = U$, $|D| < |U|$. We claim that $D = 1$. Suppose not. Now $U^tDU^rB = U^s$ whence $t < s$ and $DU^rB = U^{s-t}$. Hence $D(DF)^rB = (DF)^{s-t}$. Since $r \in \mathbf{Z}^+$, $|(DF)^{s-t}| \geq |D^2F| > |DF|$. Hence $s-t \geq 2$. So $DDF(DF)^{r-1}B = (DF)^2(DF)^{s-t-2}$. Hence $D^2F = DFD$ implying $DF = FD = U$. By Lemma 1.1 (v) $F = 1$ whence $|D| = |U|$, a contradiction. So $D = 1$ and $A \in \langle\langle U\rangle\rangle^1$. By (iii) $B \in \langle\langle U\rangle\rangle^1$.

(v) As $A_1 \neq A_2$ we can assume by symmetry that $|A_1| < |A_2|$. So for some $C \in \mathcal{F}$, $A_2 = A_1C$, $|C| = |A_2| - |A_1| > 0$ and $\omega B_1 = C\omega B_2$. Now $|B_1| = |B_2| + |C| > |B_2|$. So $B_1 = DB_2$ for some $D \in \mathcal{F}$ whence $\omega D = C\omega$. We are now done by Lemma 1.1 (vi).

(vi) $\omega = u^k$ for some $k \in \mathbf{Z}^+$. Evidently the letters appearing in u are the same as those appearing in $u^k = \omega$.

(vii) For some $S, T \in \mathcal{F}^1$, $SUxUT \in \langle\langle U\rangle\rangle^1$. By (iv), $xUT \in \langle\langle U\rangle\rangle$ and again by (iv) $x \in \langle\langle U\rangle\rangle^1$.

Lemma 1.3. *Let $\omega_1, \omega_2, \omega_3 \in \mathcal{F}$.*

(i) *Suppose $\omega_1 \to \omega_2$. Then there exists $j \in \mathbf{Z}^+$, $u, v \in \mathcal{F}^1$ such that $|u|, |v| < |\omega_2|$, $\omega_2^j = u\omega_1 v$. If further $\omega_1 \nmid \omega_2$ then there exist $x, y \in \mathcal{F}^1$, $k \in \mathbf{N}$ such that $|x|, |y| < |\omega_2|$, $x \mid_f \omega_2$, $y \mid_i \omega_2$ and $\omega_1 = x\omega_2^k y$.*

(ii) *If $\omega_1 \mid \omega_2^i$, $i \in \mathbf{Z}^+$, then for some $u, v \in \mathcal{F}^1$, $|u| < |\omega_2|$ and $u\omega_1 v = \omega_2^i$.*

(iii) *$\omega_1 \mid \omega_2 \to \omega_3$ implies $\omega_1 \to \omega_3$.*

(iv) *If $|\omega_2| \geq |\omega_1^2|$ and $\omega_1^2 \to \omega_2$ then $\omega_1 \mid \omega_2$.*

(v) *Let $\omega_2 \nmid \omega_1$ and $\omega_1 \to \omega_2$. Then $\omega_1 \mid \omega_2^2$ and $|\omega_1| < 2|\omega_2|$.*

(vi) *Suppose $\omega_1 \to \omega_3$, $\omega_2 \to \omega_3$ and $|\omega_1| \geq |\omega_2| + |\omega_3|$. Then $\omega_2 \mid \omega_1$.*

Proof. (i) We have $\omega_1 \mid \omega_2^j$ for some $j \in \mathbf{Z}^+$, j minimal. So $u\omega_1 v = \omega_2^j$ for some $u, v \in \mathcal{F}^1$. If $|u| \geq |\omega_2|$ then $u = \omega_2 u_1$ for some $u_1 \in \mathcal{F}^1$ and $u_1\omega_1 v = \omega_2^{j-1}$. Since $\omega_1 \neq 1$ we get $j - 1 \in \mathbf{Z}^+$ contradicting the minimality of j. So $|u| < |\omega_2|$. Similarly $|v| < |\omega_2|$. Next assume $\omega_1 \nmid \omega_2$. Now if $u = 1$, then $u'\omega_1 v = \omega_2^{j+1}$ where $u' = \omega_2$. It follows that for some $u', v' \in \mathcal{F}^1$ and $l \in \mathbf{Z}^+$, $u'\omega_1 v' = \omega_2^l$ and $0 < |u'| \leq |\omega_2|$, $0 < |v'| \leq |\omega_2|$. Since $\omega_1 \nmid \omega_2$, $l \geq 2$. For some $x, y \in \mathcal{F}^1$, $u'x = \omega_2 = yv'$. Hence $|x| < |\omega_2|$, $|y| < |\omega_2|$, $x \mid_f \omega_2$, $y \mid_i \omega_2$ and $\omega_1 = x\omega_2^{l-2}y$.

(ii) If $\omega_1 \mid \omega_2^j$, $j \in \mathbf{Z}^+$ and j minimal then $j \leq i$ and as above $u\omega_1 v = \omega_2^j$ for some $u, v \in \mathcal{F}^1$ with $|u|, |v| < |\omega_2|$. Hence $u\omega_1 v' = \omega_2^i$ where $v' = v\omega_2^{i-j}$.

(iii) We have $\omega_1 \mid \omega_2 \mid \omega_3^j$ for some $j \in \mathbf{Z}^+$ implying $\omega_1 \to \omega_3$.

(iv) By (i) there exist $u, v \in \mathcal{F}^1$, $j \in \mathbf{Z}^+$ such that $|u|, |v| < |\omega_2|$ and $\omega_2^j = u\omega_1^2 v$. If $|u\omega_1| \leq |\omega_2|$, then $u\omega_1 \mid \omega_2$ and we are done. So let $|\omega_2| < |u\omega_1|$. So $|u| < |\omega_2| < |u\omega_1|$ and $\omega_2 \mid_i u\omega_1$. Since $\omega_2 = u\omega_1$ is ruled out, Lemma 1.2 implies that for some $D, F \in \mathcal{F}^1$, $|D| < |\omega_1|, \omega_2 = uD$ and $\omega_1 = DF$. Hence $uDF\omega_1 v = uD\omega_2^{j-1}$ and $F\omega_1 v = \omega_2^{j-1}$. Since $|\omega_2| \geq |\omega_1^2| \geq |F\omega_1|$ we have that $j - 1 \geq 1$ and $F\omega_1 \mid_i \omega_2$. So $\omega_1 \mid \omega_2$.

(v) By (i) $u\omega_1 v = \omega_2^j$ for some $u, v \in \mathcal{F}^1$; $j \in \mathbf{Z}^+$ and $u, v \in \mathcal{F}^1$ with $|u|, |v| < |\omega_2|$. We claim that $j \leq 2$. So assume $j > 2$. Hence $\omega_2 = uD = Fv$ for some $D, F \in \mathcal{F}$. Thus

$\omega_1 = D\omega_2^{j-2}F$ implying $\omega_2 | \omega_1$ a contradiction. Hence $j \leqslant 2$. So $\omega_1 | \omega_2^2$ and $|\omega_1| \leqslant |\omega_2^2|$. If $|\omega_1| = |\omega_2^2|$ then $\omega_1 = \omega_2^2$ and $\omega_2 | \omega_1$ a contradiction. So $|\omega_1| < |\omega_2^2| = 2|\omega_2|$.

(vi) By (i) there exist $a,b,u,v \in \mathcal{F}^1$, $j,l \in \mathbf{Z}^+$ such that $|a|,|b|,|u|,|v| < |\omega_3|$ and $a\omega_1 b = \omega_3^j$, $u\omega_2 v = \omega_3^l$. If $|u| < |a|$, then $\omega_3 u\omega_2 v = \omega_3^{l+1}$ and $|a| < |\omega_3 u| < |a| + |\omega_3|$. Thus there exist $m \in \mathbf{Z}^+$, $u' \in \mathcal{F}^1$ such that $u'\omega_2 v = \omega_3^m$ and $|a| \leqslant |u'| \leqslant |a| + |\omega_3|$. Hence $u'\omega_2 \mid_1 \omega_3^{m+j}$ and $a\omega_1 \mid_1 \omega_3^{m+j}$. So for some $c,d \in \mathcal{F}^1$, $u'\omega_2 c = a\omega_1 d = \omega_3^{m+j}$. We have $u' = ax$ for some $x \in \mathcal{F}^1$. Hence $|x| \leqslant |\omega_3|$ and $x\omega_2 c = \omega_1 d$. Since $|\omega_1| \geqslant |\omega_2| + |\omega_3| \geqslant |x| + |\omega_2|$ we get $x\omega_2 \mid_i \omega_1$ implying $\omega_2 | \omega_1$.

Definition. Let $\omega_1, \omega_2 \in \mathcal{F}$.

(1) $\omega_1 \equiv \omega_2$ if $\omega_1 = AB$, $\omega_2 = BA$ for some $A,B \in \mathcal{F}^1$.

(2) $\omega_1 \sim \omega_2$ if $\omega_1 \in \langle\langle U \rangle\rangle$ and $\omega_2 \in \langle\langle V \rangle\rangle$ for some $U, V \in \mathcal{F}$ with $U \equiv V$.

(ii), (iv) and (v) of the next lemma are variations of observations in [6]. (v) below in its present form was pointed out to the author by Professor Boris Schein in response to [9; Problem 5.8].

Lemma 1.4. *Let $U, V, \omega \in \mathcal{F}$.*

(i) *$U \equiv V$ if and only if $U \to V$ and $|U| = |V|$.*

(ii) *If $U \equiv V$ then U is primitive if and only if V is primitive.*

(iii) *$U \to V \sim \omega$ implies $U \to \omega$.*

(iv) *Suppose U, V are primitive, $|U| \geqslant |V|$ and $U^2 \to V$. Then $U \equiv V$.*

(v) *$U \sim V$ if and only if $U^i \to V$ for all $i \in \mathbf{Z}^+$.*

(vi) *The relations $\equiv$ and $\sim$ are equivalence relations on $\mathcal{F}$ and $\equiv \subseteq \sim \subseteq$ —.*

(vii) *If U and V are not primitive and U—V then $U \sim V$.*

(viii) *If $U \sim V$ and $U \not\equiv V$ then $U | V$ or $V | U$.*

(ix) *$U \sim V$ if and only if $U^i \equiv V^j$ for some $i,j \in \mathbf{Z}^+$.*

Proof. We first note that $\equiv$ and $\sim$ are trivially symmetric and reflexive.

(i) Suppose $U \to V$, $|U| = |V|$. By Lemma 1.3 (i), there exist $u,v \in \mathcal{F}^1$, $j \in \mathbf{Z}^+$ such that $uUv = V^j$, $|u|,|v| < |V|$. So $|V^j| < |VUV| = |V^3|$. Thus $j \leqslant 2$. If $j = 1$, then $U = V$. So let $uUv = V^2$. Hence $V = uD = Fv$ for some $D,F \in \mathcal{F}$ implying $U = DF$. Now $|F| = |U| - |D| = |V| - |D| = |u|$. So $F = u$ and $V = FD$. Hence $U \equiv V$. The converse is trivial.

(ii) $U = AB$, $V = BA$ for some $A,B \in \mathcal{F}^1$. Suppose U is not primitive and $U = C^j$ for some $C \in \mathcal{F}$, $j \in \mathbf{Z}^+$, $j \geqslant 2$. Hence $AB = C^j$ whence $A = C^kD$, $C = DF$ for some $D,F \in \mathcal{F}^1$, $k \in \mathbf{N}$ with $|D| < |C|$, $k \leqslant j$. If $j = k$, then $B = 1$ and $U = V$. So let $k < j$. Then $B = F(DF)^{j-k-1}$. So $V = BA = F(DF)^{j-k-1}(DF)^kD = (FD)^j$. So V is not primitive.

(iii) $U \to V$ implies $U | V^i$. $V \sim \omega$ implies $V^i \to \omega$. By Lemma 1.3 (iii), $U \to \omega$.

(iv) Since $|U| \geqslant |V|$, we can write $U = WD$ for some $W,D \in \mathcal{F}^1$ with

$|W| = |V|$. Since $W|U^2$, $U^2 \to V$ we obtain $W \to V$. By (i) $W \equiv V$. By (ii) W is primitive. Since $V \sim W$, and $U^2 \to V$, (iii) implies $U^2 \to W$. Since $WDW \mid U^2$ we get $WDW \to W$. By Lemma 1.2 (vii) $D \in \langle\langle W \rangle\rangle^1$. So $U = WD \in \langle\langle W \rangle\rangle$. Since U is primitive we get $U = W \equiv V$.

(v) Let $j \in \mathbf{Z}^+$ and set $m = 2j|V|$. So $U^m \to V^j$, $|U^m| \geq m = 2|V^j|$. By Lemma 1.3 (v), $V^j | U^m$. Hence $V^j \to U$ for all $j \in \mathbf{Z}^+$. Now let $U = S^j$, $V = T^k$, $S, T \in \mathcal{F}$, S, T primitive, $j,k \in \mathbf{Z}^+$. Then $S^2 \mid U^2 \to V \sim T$. Hence $S^2 \to T$. Also $T^2 \mid V^2 \to U \sim S$ whence $T^2 \to S$. By (iv) $S \equiv T$. Hence $U \sim V$. The converse is trivial.

(vi) That $\equiv \subseteq \sim \subseteq$ — is obvious. We are to show $\equiv$ and $\sim$ are transitive. First suppose $U \equiv V \equiv \omega$. Then $|U| = |\omega|$ and by (iii) $U \to \omega$. Hence by (i) $U \equiv \omega$. Next let $U \sim V \sim \omega$. By (v) $U^i \to V \sim \omega$ for all $i \in \mathbf{Z}^+$. By (iii) $U^i \to \omega$ for all $i \in \mathbf{Z}^+$. Now by (v) $U \sim \omega$.

(vii) Let $U = U_1^i$, $V = V_1^j$, U_1, V_1 primitive, $i,j \in \mathbf{Z}^+$, $i,j \geq 2$. So $U_1^2 | U \to V \sim V_1$ implying $U_1^2 \to V_1$. Similarly $V_1^2 \to U_1$. By (iv) $U_1 \equiv V_1$ implying $U \sim V$.

(viii) For some $A,B \in \mathcal{F}^1$, $i,j \in \mathbf{Z}^+$, $U = (AB)^i$ and $V = (BA)^j$. Since $U \not\equiv V$, $i \neq j$. By symmetry let $i < j$. Then $U = (AB)^i | B(AB)^i A = (BA)^{i+1} | (BA)^j = V$. Hence $U | V$.

(ix) Suppose $U \sim V$. Then for some $A,B \in \mathcal{F}^1$, $i,j \in \mathbf{Z}^+$, $U = (AB)^j$, $V = (BA)^i$. So $U^i = (AB)^{ij} \equiv (BA)^{ij} = V^j$. Conversely let $U^i \equiv V^j$. Then $U \sim U^i \equiv V^j \sim V$ and by (vi) $U \sim V$.

2. Irreducible sequences and polygons

Definition. Let $\mathcal{F}$ be a free semigroup on some non-empty set and let $\mathcal{L}$ be a finite non-empty subset of $\mathcal{F}$.

(1) $\mathcal{L}$ is a line if $\mathcal{L}$ consists of distinct points $u_1, \ldots, u_t$ ($t \in \mathbf{Z}^+$) such that $u_1 — u_2 — \ldots — u_t$ and $u_i \not— u_{i+j}$ for $i, i+j \in \{1, \ldots, t\}$, $j \geq 2$.

(2) $\mathcal{L}$ is a union of lines if $\mathcal{L}$ is a disjoint union of non-empty subsets $\mathcal{L}_j$ ($j = 1, \ldots, m$) such that each $\mathcal{L}_j$ is a line and $a \in \mathcal{L}_j$, $b \in \mathcal{L}_k$, $j \neq k$ implies that $a \not— b$.

(3) By a polygon (n-gon) $\mathcal{P}$ in $\mathcal{F}$ we mean a set of n distinct points $u_1, \ldots, u_n \in \mathcal{F}$, $n \geq 3$, such that $u_1 — u_2 — \ldots — u_n — u_1$. $\mathcal{P}$ is irreducible if $1 \leq i < i+j \leq n$, $j \geq 2$, $u_i — u_{i+j}$ implies $i = 1$ and $i + j = n$. By a polygon through $\mathcal{L}$ we mean a polygon whose set of vertices contains $\mathcal{L}$.

[9; Conjecture 4.10] is false as we will see in Example 2.3. Following theorem is a weaker form of that conjecture. The following also strengthens [9; Corollary 4.9].

Theorem 2.1. *Let $\omega_1, \omega_2 \in \mathcal{C}$, $\xi \in \mathbf{Z}^+$, $\omega_1 \not— \omega_2$. Then there exists $k \in \mathbf{Z}^+$ such that for every $l \geq k$ there exists an irreducible sequence $\langle U_i \rangle_{i=1}^l$ of length l between ω_1 and ω_2 such that for every $i = 1, \ldots, l$, either $\omega_1^\xi \mid U_i$ or $\omega_2^\xi \mid U_i$.*

Proof. Either $\omega_1 \nrightarrow \omega_2$ or $\omega_2 \nrightarrow \omega_1$. By symmetry we assume $\omega_2 \nrightarrow \omega_1$. By Lemma 1.2 (vi) the primitive root U of ω_1 lies in $\mathcal{C}$. Now $\omega_1 = U^t$ for some $t \in \mathbf{Z}^+$. $\omega_2 \nrightarrow \omega_1$ implies $\omega_2 \nrightarrow U$. This implies that $\mathcal{C}$ is a free content on more than one letter and hence that $|U| > 1$. Let U start with the letter ϵ. So $|\epsilon| = 1$. We claim that $U\epsilon U \nrightarrow U$ for otherwise by Lemma 1.2 (vii), $\epsilon \in \langle\langle U \rangle\rangle^1$, a contradiction. Hence

(1) $\qquad U\epsilon U \nrightarrow U$.

Let $j \in \mathbf{Z}^+$, $j \geqslant 2$. We claim that $U^{2^{j+1}} \nrightarrow U^{2^{j-1}}\epsilon U^{2^{j-1}}$. For suppose $U^{2^{j+1}} \to U^{2^{j-1}}\epsilon U^{2^{j-1}}$. Then $U\epsilon U \to U^{2^{j-1}}\epsilon U^{2^{j-1}}$ and $|U\epsilon U| + |U^{2^{j-1}}\epsilon U^{2^{j-1}}| \leqslant |U^{2^{j+4}}| \leqslant |U^{2^{j+1}}|$. By Lemma 1.3 (vi) $U\epsilon U \mid U^{2^{j+1}}$ contradicting (1). So

(2) $\qquad U^{2^{j+1}} \nrightarrow U^{2^{j-1}}\epsilon U^{2^{j-1}}$ for any $j \in \mathbf{Z}^+$, $j \geqslant 2$.

(1) and (2) imply that the following is a line for each $n \in \mathbf{Z}^+$, $n \geqslant 2$

(3) $\qquad U — U^{2^n}\epsilon — U^{2^{n-1}}\epsilon U^{2^{n-1}} — \ldots — U^2\epsilon U^2$ $(n \in \mathbf{Z}^+,\ n \geqslant 2)$.

Since $\omega_2 \nrightarrow U$, Lemma 1.4 tells us that $U^m \nrightarrow \omega_2$ for some $m \in \mathbf{Z}^+$, $m \geqslant t\xi$. Hence

(4) $\qquad U^{2^\alpha} \nrightarrow \omega_2,\quad U^{2^\alpha}\epsilon U^{2^\alpha} \nrightarrow \omega_2$ for any $\alpha \in \mathbf{Z}^+$, $\alpha \geqslant m \geqslant t\xi$.

Let $\omega_3 = U^{2^m}\epsilon U^{2^m}$. By [9; Theorem 4.8] there exists an irreducible sequence $\langle V_i \rangle_{i=1}^{\beta}$ $(\beta \in \mathbf{Z}^+)$ in $\mathcal{C}$ between ω_3 and ω_2 such that

(5) $\qquad \omega_3 \mid V_i$ or $\omega_2^\xi \mid V_i$ $(i = 1, \ldots, \beta)$.

Now $\omega_2 \nrightarrow U$ and by (1) $\omega_3 \nrightarrow U$. Consequently

(6) $\qquad V_i \nrightarrow U$ $(i = 1, \ldots, \beta)$.

By Lemma 1.4, there exists $\gamma \in \mathbf{Z}^+$ such that $\gamma \geqslant m$ and $U^{2^\gamma} \nrightarrow V_i$ $(i = 1, \ldots, \beta)$. Hence

(7) $\qquad U^{2^\delta} \nrightarrow V_i$ $(i = 1, \ldots, \beta)$ for any $\delta \in \mathbf{Z}^+$, $\delta \geqslant \gamma \geqslant m$.

Choose j between 1 and β largest so that V_j—-$U^{2^\nu}\epsilon U^{2^\nu}$ for some $\nu \in \mathbf{Z}^+$, $\nu \geqslant m$. By (7) $\nu \leqslant \gamma$. Choose ν maximal. So $\gamma \geqslant \nu \geqslant m$ and

(8) $\qquad$ If $k, \theta \in \mathbf{Z}^+$, $j \leqslant k \leqslant \beta$, $\theta \geqslant \nu$ then $U^{2^\theta}\epsilon U^{2^\theta}$—-$V_k$ if and only if $j = k$ and $\theta = \nu$.

By (3), (4), (5), (6), (7), (8) and the fact that $\omega_2 \nrightarrow U$, we have that the following is a line for any $n \in \mathbf{Z}^+$, $n > \gamma$.

(9) $\qquad U — U^{2^n}\epsilon — U^{2^{n-1}}\epsilon U^{2^{n-1}} — \ldots — U^{2^\nu}\epsilon U^{2^\nu} — V_j — \ldots — V_\beta — \omega_2$

(for any $n > \gamma$, $n \in \mathbf{Z}^+$).

If $n > \gamma$ then since $\gamma \geqslant m \geqslant \xi t$ we see that $\omega_1 = U^t \mid U^{2^n}\epsilon$. Since $U \sim \omega_1$ we see that $U^{2^n}\epsilon — \omega_1$. Moreover $\omega \in \mathcal{C}$, $\omega — \omega_1 = U^t$ implies $\omega — U$. It follows that the following is a line for any $n > \gamma$:

$$\omega_1 — U^{2^n}\epsilon — U^{2^{n-1}}\epsilon U^{2^{n-1}} — ... — U^{2^\nu}\epsilon U^{2^\nu} — V_j — ... — V_\beta — \omega_2 \quad (n > \gamma). \tag{10}$$

Since $\nu \geqslant m \geqslant \xi t$, (5) tells us that every word other than ω_1, ω_2 in the above line has as a segment either ω_2^ξ or $U^m = \omega_1^\xi$. Moreover for each $n > \gamma$, (10) yields an irreducible sequence between ω_1 and ω_2 of length $n - \nu + \beta - j + 2$. Since ν, β and j are fixed, the theorem is proved.

Remark 2.2. The above proof is constructive except for the use of [9; Theorem 4.8] of which we had given a semigroup theory proof. However, it is possible to give a constructive proof of that theorem by using (1) on [9; p. 310] and an idea of John Shafer (see [9; Remark 3.4 (ii)]).

Example 2.3. We now give some examples showing that [9; Conjecture 4.10] is false. Let $\mathcal{F} = \mathcal{F}(A,B)$. We have $ABA \not— BAB$,

$$ABA — AB — BAB\,.$$

Also for $s \in \mathbf{Z}^+$, $s \geqslant 2$, we have the line

$$ABA — B(A^2B)^s — (A^2B)B(A^2B)^{s-1} — ...$$
$$... — (A^2B)B(A^2B) — AB^2A — AB^2 — BAB\,.$$

Thus the minimal sequence between ABA and BAB has length 1. Also we have produced irreducible sequences of length t for every $t \geqslant 4$. On the other hand it is not hard to show that there exist no irreducible sequences between ABA and BAB of length 2 or 3.

Also $AB^2AB \not— BAB^2$ and,

$$AB^2AB — AB^2 — BAB^2\,.$$

Further for $s \in \mathbf{Z}^+$, $s \geqslant 2$ we have the line,

$$AB^2AB — B[(AB)^2B]^s — [(AB)^2B]B[(AB)^2B]^{s-1} — ...$$
$$... — [(AB)^2B]B[(AB)^2B] — (AB)B^2(AB) — BAB^2\,.$$

So the minimal sequence between AB^2AB and BAB^2 has length 1. We have produced irreducible sequences between AB^2AB and BAB^2 of length t for every $t \geqslant 3$. On the other hand, it can be shown that there exists no irreducible sequence between AB^2AB and BAB^2 of length 2.

Problem 2.4. Let $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_1 \not— \omega_2$. Study the initial behavior of the lengths of the irreducible sequences between ω_1 and ω_2.

Lemma 2.5. *Let $\mathcal{L}$ be a finite non-empty subset of $\mathcal{C}$ and suppose that $\mathcal{L}$ is a union of lines. Correspondingly let $\mathcal{L}$ be the disjoint union of $\mathcal{L}_i$ ($i = 1, ..., m$), $\mathcal{L}_i = \{u_{\alpha_i}—...—u_{\beta_i}\}$. Assume further that for any $i \in \{1, ..., m\}$ and $u \in \mathcal{L}$, $u \neq u_{\alpha_i}$ implies $u \not\rightarrow u_{\alpha_i}$ and $u \neq u_{\beta_i}$ implies $u \not\rightarrow u_{\beta_i}$. Then $\mathcal{L} \subseteq \mathcal{M} \subseteq \mathcal{C}$ for some finite line $\mathcal{M} = \{x_1—...—x_s\}$ such that $y \in \mathcal{M}$, $y \neq x_1$ implies $y \not\rightarrow x_1$ and $y \neq x_s$ implies $y \not\rightarrow x_s$.*

Proof. By the very definition of union of lines, $u \in \mathcal{L}_i$, $v \in \mathcal{L}_j$, $i \neq j$ implies $u \not—v$. We prove the lemma by induction on m. If $m = 1$ there is nothing to prove. So let $m > 1$. Now $\mathcal{L}_1 = \{u_{\alpha_1}—...—u_{\beta_1}\}$ and $\mathcal{L}_2 = \{u_{\alpha_2}—...—u_{\beta_2}\}$. Our hypothesis implies by Lemma 1.4 that there exists $n \in \mathbf{Z}^+$ such that $\omega \in \mathcal{L}$, $\omega \neq u_{\beta_1}$ implies $u_{\beta_1}^n \not\leftrightarrow \omega$ and $\omega \neq u_{\alpha_2}$ implies $u_{\alpha_2}^n \not\leftrightarrow \omega$. Since $u_{\beta_1} \not— u_{\alpha_2}$ we have by [9; Theorem 4.8] that there exists an irreducible sequence $\langle v_j \rangle_{j=1}^t$ between u_{β_1} and u_{α_2} such that for each $j \in \{1, ..., t\}$ either $u_{\beta_1}^n \mid v_j$ or $u_{\alpha_2}^n \mid v_j$. Set $v_0 = u_{\beta_1}$ and $v_{t+1} = u_{\alpha_2}$. We thus have the line,

$$v_0—v_1—...—v_t—v_{t+1}\ .$$

Now it cannot be that $t = 1$ for otherwise $u_{\alpha_2}^n \mid v_1 \rightarrow v_0 = u_{\beta_1}$ or $u_{\beta_1}^n \mid v_1 \rightarrow v_{t+1} = u_{\alpha_2}$ a contradiction. Hence $t > 1$. Now suppose for some $1 \leqslant j \leqslant t$, $v_0 \sim v_j$. Then $v_0—v_j$ and $j = 1$. So $v_2—v_1 \sim v_0$ implying $v_2 \rightarrow v_0$. So $v_0 \not\leftrightarrow v_2$ and $u_{\beta_1}^n \nmid v_2$. Since $2 \leqslant t$ we get $u_{\alpha_2}^n \mid v_2 \rightarrow v_0 = u_{\beta_1}$ implying $u_{\alpha_2}^n \rightarrow u_{\beta_1}$ a contradiction. So $v_0 \not\rightarrow v_j$ for $1 \leqslant j \leqslant t$. Similarly $v_{t+1} \not\rightarrow v_j$ $(1 \leqslant j \leqslant t)$. Next let $\omega \in \mathcal{L}$, $\omega \neq u_{\beta_1}$, $\omega \neq u_{\alpha_2}$. Then for v_j, $j \in \{1, ..., t\}$ either $u_{\beta_1}^n \mid v_j$ or $u_{\alpha_2}^n \mid v_j$ whereby $v_j \not\leftrightarrow \omega$. We have thus shown,

$$u_{\beta_1} \not\rightarrow v_j\,,\quad u_{\alpha_2} \not\rightarrow v_j\,;\quad v_j \not\leftrightarrow \omega \quad \text{for} \quad \omega \in \mathcal{L},\quad \omega \neq u_{\beta_1},\quad \omega \neq u_{\alpha_2}$$
$$(j = 1, ..., t)\,.$$

In particular $\{v_1, ..., v_t\} \cap \mathcal{L} = \emptyset$ and whether or not $\alpha_1 = \beta_1$ or $\alpha_2 = \beta_2$ we have,

$$u_{\alpha_1} \not\rightarrow v_j\,,\quad u_{\beta_2} \not\rightarrow v_j \qquad (j = 1, ..., t)\,.$$

We further have the line

$$\mathcal{L}'_1 = \{u_{\alpha_1}—...—u_{\beta_1}—v_1—...—v_t—u_{\alpha_2}—...—u_{\beta_2}\}\,.$$

By the preceding discussion, $\mathcal{L}'_1 \cup \mathcal{L}_3 \cup ... \cup \mathcal{L}_m$ is a union of lines, satisfies the hypothesis of the theorem, and $\mathcal{L} \subseteq \mathcal{L}'_1 \cup ... \cup \mathcal{L}_m$. We are therefore done by our induction hypothesis on m.

The following theorem generalizes [9; Corollary 5.10].

Theorem 2.6. *Let $\mathcal{L}$ be a finite non-empty subset of $\mathcal{C} = \mathcal{C}(A_1, ..., A_t)$, $t \geqslant 2$. Suppose further that for any $u, v \in \mathcal{L}$, $u \equiv v$ implies $u = v$. Then the following are equivalent.*

(i) *$\mathcal{L}$ is a union of lines and for any $u, v \in \mathcal{L}$, $u \sim v$ implies $u = v$.*

(ii) *There exists $k \in \mathbf{Z}^+$ such that for every $l \in \mathbf{Z}^+$, $l \geqslant k$, there exists an irreducible l-gon in $\mathcal{C}$ through $\mathcal{L}$.*

(iii) *There exists an irreducible n-gon through $\mathcal{L}$ for some $n \in \mathbf{Z}^+$ such that $n \geqslant 4$ and $n \geqslant |\mathcal{L}| + 1$.*

Proof. (i) $\Rightarrow$ (ii). By Lenna 2.5 $\mathcal{L} \subseteq \mathcal{M}$ where $\mathcal{M} = \{u_1 — ... — u_\alpha\}$, $u_1 \nrightarrow u_i$ for $i \neq 1$ and $u_\alpha \nrightarrow u_i$ for $i \neq \alpha$. If $\alpha = 1$ we are done by [9; Theorem 5.4]. So Let $\alpha \geqslant 2$. By Lemma 1.4 there exists $\gamma \in \mathbf{Z}^+$ such that $u_1^\gamma \nrightarrow u_i$ for $i \neq 1$ and $u_\alpha^\gamma \nrightarrow u_i$ for $i \neq \alpha$. So $u_1^\gamma \not\!— u_\alpha^\gamma$. Theorem 2.1 tells us that there exists $k \in \mathbf{Z}^+$ such that for every $l \in \mathbf{Z}^+$, $l \geqslant k$, there exists an irreducible sequence, say $\langle V_j \rangle_{j=1}^l$, between u_1^γ and u_α^γ such that for each $j \in \mathbf{Z}^+$, $1 \leqslant j \leqslant l$, either $u_1^\gamma | V_j$ or $u_\alpha^\gamma | V_j$. Evidently $\langle V_j \rangle_{j=1}^l$ is a sequence between u_1 and u_α and $\{V_j | j = 1, ..., l\}$ is a line. Now suppose for some $i, j \in \mathbf{Z}^+$, $1 \leqslant i \leqslant \alpha$, $1 \leqslant j \leqslant l$, $u_i — V_j$. Now either $u_1^\gamma | V_j$ or $u_\alpha^\gamma | V_j$. First assume $u_1^\gamma | V_j$. Then $u_1^\gamma \to u_i$ which implies $i = 1$. So $u_1 — V_j$. Since $u_1^\gamma | V_j$ we get $u_1^\gamma — V_j$ whence $j = 1$. Similarly $u_\alpha^\gamma | V_j$ implies $i = \alpha$ and $j = l$. So we have the following irreducible polygon

$$u_1 — u_2 — ... — u_\alpha — v_l — ... — v_1 — u_1 .$$

We have thus produced an irreducible $\alpha + l$-gon through $\mathcal{M}$ and hence $\mathcal{L}$ for every $l \geqslant k$. Since k and α are fixed, we are done.

(ii) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (i). Since any non-empty proper subset of the set of vertices of an irreducible polygon is a union of lines, it follows that $\mathcal{L}$ is a union of lines. Now suppose $u, v \in \mathcal{L}$, $u \neq v$, $u \sim v$. Since by hypothesis $u \not\equiv v$, it follows by Lemma 1.4 that $u | v$ or $v | u$. By symmetry assume $u | v$. Then for any $\omega \in \mathcal{C}$, $v — \omega$ implies $u \to \omega$ (since $u | v$) and $\omega \to u$ (since $v \sim u$) whereby $u — \omega$. Thus $u — v$; $v — \omega$ implies $u — \omega$ for any $\omega \in \mathcal{C}$. But this contradicts the fact that u and v are distinct vertices of an irreducible n-gon, $n \geqslant 4$.

Remark 2.7. Suppose $\mathcal{L}$ is a finite non-empty subset of $\mathcal{C}$. Suppose $\mathcal{L} = \{u_1 — ... — u_\alpha\}$ is a line, $\alpha \in \mathbf{Z}^+$. Then it can be shown that (ii) of the above theorem holds for $\mathcal{L}$ if and only if there exist $v, \omega \in \mathcal{C}$ such that the following is a line,

$$v — u_1 — ... — u_\alpha — \omega .$$

Thus if $\mathcal{L} = \{u_1, u_2\}$, $u_1 \equiv u_2$, $u_1 \neq u_2$ then (ii) of Theorem 2.6 holds for $\mathcal{L}$ if and only if there exist $v, \omega \in \mathcal{C}$ such that $v — u_1$, $u_2 — \omega$, $v \not\!— u_2$, $\omega \not\!— u_1$ and $v \not\!— \omega$. This may or may not hold. In $\mathcal{C}(A,B)$ it holds for $\{A^2B^2, B^2A^2\}$, $\{ABA, A^2B\}$ and $\{A^2BAB, ABA^2B\}$. However, it fails for $\{AB, BA\}$, $\{A^2B, BA^2\}$ and $\{ABA^2, A^2BA\}$. The exact necessary and sufficient conditions, in terms of the structure of u_1 and u_2 are not known to us. We will see in Section 6 that the situation is simpler if we replace the positive integer exponents in the free semigroup by positive rational exponents.

3. Induced subgraphs

By a *graph* we will mean in this paper a pair $(\Gamma, —)$ where Γ is a non-empty set and — is a symmetric and reflexive relation on Γ. We caution that graph theorists often use the term (undirected) 'graph' to mean a set with a symmetric irreflexive relation. But of course this difference is only a technical one. If $(\Gamma, —)$ and $(\Lambda, —)$ are graphs then $(\Lambda, —)$ is an *induced subgraph* of $(\Gamma, —)$ if and only if $\Lambda \subseteq \Gamma$ and for all $x_1, x_2 \in \Lambda$, $x_1 — x_2$ in Λ if and only if $x_1 — x_2$ in Γ. On the other hand, starting with just a non-empty subset Λ of Γ we make Λ into an induced subgraph of Γ by defining, for $y_1, y_2 \in \Lambda$, $y_1 — y_2$ in Λ if and only if $y_1 — y_2$ in Γ. At times if Λ is only isomorphic to an induced subgraph of Γ we might still say Λ is an induced subgraph of Γ provided there is no ambiguity. A graph Γ is *connected* if for any $x, y \in \Gamma$ there exist $t \in \mathbf{Z}^+$ and $x_1, ..., x_t \in \Gamma$ (not necessarily distinct) such that $x — x_1 — ... — x_t — y$. Note that if $x — y$ then $x — x — y$. The connected components of Γ are the maximal connected induced subgraphs of Γ; these partition Γ. As noted in Section 1, the connected components of the archimedean graph $\mathcal{F}(X)$ are $\mathcal{C}(Y)$, Y a finite non-empty subset of X. Moreover the archimedean relation on $\mathcal{C}(Y)$ as a semigroup coincides with the restriction of the archimedean relation on $\mathcal{F}(X)$ to $\mathcal{C}(Y)$. A graph $(\Gamma, —)$ is *complete* if and only if for any $x, y \in \Gamma$, $x — y$. A graph $(\Gamma, —)$ is a *tree* if it is connected and for any distinct $x_1, ..., x_t \in \Gamma$, $t \geqslant 3$, it is not true that $x_1 — x_2 — ... — x_t — x_1$.

Theorem 3.1. *Let $m, n \in \mathbf{Z}^+$, $m, n \geqslant 2$. Then the archimedean graph of $\mathcal{F}(A_1, ..., A_n)$ is isomorphic to an induced subgraph of $\mathcal{C}(A_1, ..., A_m) \subseteq \mathcal{F}(A_1, ..., A_m)$. In particular the induced subgraphs of $\mathcal{F}(A_1, ..., A_n)$ are, within isomorphism, the same as the induced subgraphs of $\mathcal{C}(A_1, ..., A_m)$.*

Proof. We first show that $\mathcal{F}(A_1, ..., A_n)$ is an induced subgraph of $\mathcal{C}(A,B)$. Set $u_j = ABA^{j+1}B^{j+1}$ $(j = 1, ..., n)$. Since $u_j \nleq_{\mathrm{i}} u_k$ for $j \neq k$, it is well-known [2; Theorem 9.1] that $\mathcal{E} = \langle\langle u_1, ..., u_n \rangle\rangle$, the semigroup generated by $u_1, ..., u_n$, is a free semigroup on $\{u_1, ..., u_n\}$. Hence as semigroups $\mathcal{F}(A_1, ..., A_n)$ is isomorphic to $\mathcal{E}$. So the corresponding archimedean graphs are isomorphic. Also $\mathcal{E} \subseteq \mathcal{C}(A,B)$. Thus what we must show now is that the archimedean relation on $\mathcal{E}$ as a semigroup coincides with the restriction of the archimedean relation of $\mathcal{F}(A,B)$ to $\mathcal{E}$. First let $\omega \in \mathcal{E}$ and $x, y \in \mathcal{F}(A,B)^1$ such that $x\omega y \in \mathcal{E}$. We claim that then $x, y \in \mathcal{E}^1$. For suppose not and choose a counter-example with $|x\omega y|$ minimal. Now either $x \notin \mathcal{E}^1$ or $y \notin \mathcal{E}^1$. First let $x \notin \mathcal{E}^1$. In $\mathcal{E}$ let $x\omega y$ start with u_j and ω start with u_k. So $\omega = u_k\omega_1$ and $x\omega y = u_j\omega_2$ for some $\omega_1, \omega_2 \in \mathcal{E}^1$. Hence $xu_k\omega_1 y = u_j\omega_2$. If $u_j \leq_{\mathrm{i}} x$ then $x = u_jx'$ whence $x'\omega y = \omega_2 \in \mathcal{E}$. By minimality of $|x\omega y|$, $x' \in \mathcal{E}^1$. Thus $x = u_jx' \in \mathcal{E}$, a contradiction. So $|x| < |u_j|$. Thus

$$(11) \qquad xABA^{k+1}B^{k+1}\omega_1 y = ABA^{j+1}B^{j+1}\omega_2, \qquad j, k \in \mathbf{Z}^+;$$

$$|x| < |ABA^{j+1}B^{j+1}|.$$

Hence $|xA| \leqslant |ABA^{j+1}B^{j+1}|$. Since obviously $xA \neq ABA^{j+1}B^{j+1}$ we get $|xAB| \leqslant |ABA^{j+1}B^{j+1}|$ implying $xAB \mid_1 ABA^{j+1}B^{j+1}$. Since $x \neq 1$, we get $x = ABA^j$ contradicting (11). So $x \in \mathcal{C}^1$. Thus $x\omega y \in \mathcal{C}$, $x\omega \in \mathcal{C}$. Since $u_\alpha \nmid_1 u_\beta$ for $\alpha \neq \beta$ we get $y \in \mathcal{C}^1$. This contradiction shows that if $\omega_1, \omega_2 \in \mathcal{C}$, then $\omega_1 \mid \omega_2$ in $\mathcal{C}$ if and only if $\omega_1 \mid \omega_2$ in $\mathcal{F}(A,B)$. It follows that for $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_1 — \omega_2$ in $\mathcal{C}$ if and only if $\omega_1 — \omega_2$ in $\mathcal{F}(A,B)$. So we have shown that $\mathcal{F}(A_1, ..., A_n)$ is an induced subgraph of $\mathcal{C}(A,B)$. We now proceed to show that $\mathcal{C}(A,B)$ is an induced subgraph of $\mathcal{C}(A_1, ..., A_m)$. Consider the map $\varphi: \mathcal{F}(A,B)^1 \to \mathcal{F}(A_1, ..., A_m)^1$ given by $\varphi(A) = A_1$, $\varphi(B) = A_2 \ ... \ A_m$. Define $\theta : \mathcal{F}(A_1, ..., A_m)^1 \to \mathcal{F}(A,B)^1$ by $\theta(A_1) = A$, $\theta(A_2) = B$, $\theta(A_j) = 1$ for $2 \leqslant j \leqslant m$. θ is a surjective homomorphism and $\theta \circ \varphi$ is the identity map on $\mathcal{F}(A,B)^1$. It follows that φ is an injective semigroup homomorphism and that for $\omega_1, \omega_2 \in \mathcal{F}(A,B)$, $\omega_1 — \omega_2$ in $\mathcal{F}(A,B)$ if and only if $\varphi(\omega_1) — \varphi(\omega_2)$ in $\mathcal{F}(A_1, ..., A_m)$. Since φ is an injection and $\varphi(\mathcal{C}(A,B)) \subseteq \mathcal{C}(A_1, ..., A_m)$ we are done.

Remark 3.2. (1) Any induced subgraph of $\mathcal{F}(A) = \mathcal{C}(A)$ is complete where A is a single letter.

(2) The induced subgraphs of $\mathcal{F}(\mathrm{N})$ are the same as the induced subgraphs of $\mathcal{F}(A,B)$ since the archimedean graph of $\mathcal{F}(\mathrm{N})$ is isomorphic to an induced subgraph of $\mathcal{F}(A, B)$.

(3) If Γ is a graph with at most countably many connected components and if Γ is an induced subgraph of a free semigroup on any non-empty set then Γ is an induced subgraph of $\mathcal{F}(A,B)$. This is because any connected induced subgraph of any free semigroup must be contained in a free content on a non-empty finite set.

Definition. Let $(\Gamma, —)$ be a graph.

(1) Γ is contained in the free semigroup if Γ is isomorphic to an induced subgraph of the archimedean graph of some (and hence every) free semigroup on a finite set with more than one element.

(2) Let $\omega \in \mathcal{F} = \mathcal{F}(X)$, X a non-empty set. Then Γ occurs at ω (in $\mathcal{F}$) if Γ is isomorphic to an induced subgraph of $\mathcal{F}$ containing ω as a vertex.

Remark 3.3. (1) By our very definition, a graph contained in the free semigroup must be at most countable.

(2) An induced subgraph of a graph contained in the free semigroup is itself contained in the free semigroup.

Theorem 3.4. *Let $(\Gamma, —)$ be a finite graph. Then Γ is contained in the free semigroup if and only if each connected component of Γ is contained in the free semigroup.*

Proof. Let Γ_i $(i = 1, ..., m)$ be the connected components of Γ and suppose that each Γ_i is contained in the free semigroup. Choose $2m$ distinct letters $A_1, ..., A_m$, $B_1, ..., B_m$. For each i, $1 \leqslant i \leqslant m$, Γ_i is isomorphic to an induced subgraph $\mathcal{W}_i$ of

$\mathcal{F}(A_i, B_i)$. Now $u \in \mathcal{F}(A_i, B_i)$, $v \in \mathcal{F}(A_j, B_j)$, $i \neq j$ implies $u \not\longrightarrow v$. It follows that Γ is isomorphic to the induced subgraph $\mathcal{W} = \bigcup_{i=1}^{m} \mathcal{W}_i$ of $\mathcal{F}(A_1, ..., A_m, B_1, ..., B_m)$.

In Section 4 we will see that not every finite graph is contained in the free semigroup. This, for reasons that we are only now beginning to understand, is quite difficult to prove. On the other hand, it is easy to show that not every countably infinite graph is contained in the free semigroup as we now proceed to do.

Example 3.5. Let $\mathcal{F} = \mathcal{F}(A,B)$. Suppose every countably infinite graph is contained in the free semigroup. Then there must exist distinct words ω_1, ω_2, u_i $(i \in \mathbf{N})$ such that $\omega_1 \not\longrightarrow \omega_2$, $\omega_1 \longrightarrow u_i \longrightarrow u_j \longrightarrow \omega_2$ for all $i, j \in \mathbf{N}$. Since the u_i's are distinct and since in $\mathcal{F}(A,B)$ the number of words of a fixed length is finite it follows that there exists $i \in \mathbf{N}$ such that $|u_i| \geqslant 2(|\omega_1| + |\omega_2|)$. Since $u_i \longrightarrow \omega_1$ and $u_i \longrightarrow \omega_2$, Lemma 1.3 (v) implies $\omega_1 \mid u_i$ and $\omega_2 \mid u_i$. Then by Lemma 1.3 (iii) $\omega_1 \longrightarrow \omega_2$ a contradiction. So correspondingly there exists a countably infinite graph not contained in the free semigroup.

In [9; Theorem 5.4] we saw that if $\omega \in \mathcal{F}$ and ω consists of more than one letter, then every polygon occurs at ω. In contrast we now prove

Theorem 3.6. *Let $n \in \mathbf{Z}^+$. Then there exists a finite graph $(\Phi_n, \longrightarrow)$ contained in the free semigroup such that for any free semigroup $\mathcal{F}$ and any word ω in $\mathcal{F}$ of length $\leqslant n$, Φ_n does not occur at ω in $\mathcal{F}$.*

Proof. Let $m = 8n^2 + 1$ and $K = \{1, ..., m\}$. Set $\Phi_n = K \times \{1,2,3,4\}$. If $(\alpha, i), (\beta, j) \in \Phi_n$ then we define $(\alpha, i) \longrightarrow (\beta, j)$ if and only if $i = j$ or $\{i,j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\}$. We will now show that Φ_n is contained in the free semigroup. In $\mathcal{F}(A,B)$, let $\mathcal{A}_1 = \{A^\alpha B A^{2m-\alpha} | \alpha = 1, ..., m\}$, $\mathcal{A}_2 = \{A^{2m-\alpha} B A^{m+\alpha} | \alpha = 1, ..., m\}$, $\mathcal{A}_3 = \{A^\alpha B A^{3m} B A^{2m-\alpha} | \alpha = 1, ..., m\}$ and $\mathcal{A}_4 = \{A^{2m-\alpha} B A^{2m} B A^{m+\alpha} | \alpha = 1, ..., m\}$. Then $|\mathcal{A}_i| = m$ for $i = 1, ..., 4$. If $u \in \mathcal{A}_i$ and $v \in \mathcal{A}_j$, it is readily verified that $u \longrightarrow v$ if and only if $i = j$ or $\{i,j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\}$. It follows that the induced subgraph $\mathcal{A} = \bigcup_{i=1}^{4} \mathcal{A}_i$ of $\mathcal{F}(A,B)$ is isomorphic to Φ_n.

Now let $\mathcal{F}$ be a free semigroup on any non-empty set and $\omega \in \mathcal{F}$, $|\omega| \leqslant n$. We claim that Φ_n does not occur at ω in $\mathcal{F}$. For suppose otherwise. Then correspondingly there will exist disjoint subsets $\mathcal{M}_i$ $(i = 1, ..., 4)$ of $\mathcal{F}$ such that $|\mathcal{M}_i| = m = 8n^2 + 1$ $(i = 1, ..., 4)$, $\omega \in \bigcup_{i=1}^{4} \mathcal{M}_i$ and so that for $u \in \mathcal{M}_i$, $v \in \mathcal{M}_j$, $u \longrightarrow v$ if and only if $i = j$ or $\{i,j\} \in \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,1\}\}$. By symmetry we may assume $\omega \in \mathcal{M}_1$. Then for any $u \in \mathcal{M}_2$, $u \longrightarrow \omega$. Now let $\mathcal{N}' = \{u | u \in \mathcal{M}_2 \text{ and } u | \omega\}$. Then $|\mathcal{N}'| \leqslant n^2$. Let $\mathcal{N} = \mathcal{M}_2 \setminus \mathcal{N}'$. So $|\mathcal{N}| \geqslant 7n^2 + 1$. Next let $\mathcal{B}_1 = \{x | x \in \mathcal{F}^1, |x| < |\omega|, x \mid_{\mathrm{f}} \omega\}$ and $\mathcal{B}_2 = \{y | y \in \mathcal{F}^1, |y| < |\omega|, y \mid_{\mathrm{i}} \omega\}$. Then $|\mathcal{B}_1| = |\mathcal{B}_2| = |\omega| \leqslant n$. Let $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$ so that $|\mathcal{B}| \leqslant n^2$. Now for any $u \in \mathcal{N}$, $u \longrightarrow \omega$ and $u \nmid \omega$. By Lemma 1.3 (i), there exists $(x,y) \in \mathcal{B}$ and $k \in \mathbf{N}$ such that $u = x\omega^k y$. Fixing such a representation for each $u \in \mathcal{F}$ we write $k = \rho(u)$ and $(x,y) = \delta(u)$. Note that

$\psi(u) = (\rho(u), \delta(u))$ is a 1–1 function on $\mathcal{N}$. For $k \in \mathbf{N}$ let $\mathcal{D}_k = \{u \mid u \in \mathcal{N}$ and $\rho(u) = k\}$. Then δ is 1–1 on $\mathcal{D}_k$ whence $|\mathcal{D}_k| \leqslant |\mathcal{B}| \leqslant n^2$. Now let $E = \{k \mid k \in \mathbf{N}$ and $\mathcal{D}_k \neq \emptyset\}$. If $|E| \leqslant 7$ then $\mathcal{N} = \bigcup_{k \in E} \mathcal{D}_k$ and $|\mathcal{N}| \leqslant 7n^2$, a contradiction. So $|E| \geqslant 8$. It follows that there exist $j, k \in E$ such that $0 < j \leqslant k-6$. There exist $u \in \mathcal{D}_j$ and $v \in \mathcal{D}_k$. Let $\delta(u) = (x, y)$ and $\delta(v) = (x', y')$: Then $u = x\omega^j y$ and $v = x'\omega^k y'$. Since $v, u \in \mathcal{N} \subseteq \mathcal{M}_2$, v—u. Hence $\omega^k \to x\omega^j y$. Now $\omega yx\omega \to x\omega^j y$ and $|\omega yx\omega| + |x\omega^j y| \leqslant |\omega^{j+6}| \leqslant |\omega^k|$. By Lemma 1.3 (vi) $\omega yx\omega \mid \omega^k$. Now let $\omega = U^\alpha$ where $U \in \mathcal{F}$ is primitive and $\alpha \in \mathbf{Z}^+$. So $UyxU \mid \omega yx\omega \mid \omega^k = U^{k\alpha}$ and $UyxU \to U$. By Lemma 1.2 (vii), $yx \in \langle\langle U \rangle\rangle^1$. Hence $u \equiv \omega^j yx$ and $\omega^j yx \in \langle\langle U \rangle\rangle$. Thus $u \sim U \sim \omega$ implying $u \sim \omega$. We have thus produced $u \in \mathcal{M}_2$ such that $u \sim \omega$. Similarly there exists $u' \in \mathcal{M}_4$ such that $u' \sim \omega$. Consequently $u \sim u'$ implying u—u', a contradiction.

4. An example

Throughout this section $\mathcal{F}$ will denote a free semigroup on a finite non-empty set with more than one element and will remain fixed (only in this section).

Definition. (1) Let Γ_0 and Γ_1 be two graphs. Then $\Gamma_0 + \Gamma_1 = \Gamma_0 \times \{0\} \cup \Gamma_1 \times \{1\}$. If $(a,i), (b,j) \in \Gamma_0 + \Gamma_1$ then (a,i)—(b,j) if and only if $i \neq j$ or $i = j$ and a—b in Γ_i. Note that Γ_i is isomorphic to the induced subgraph $\Gamma_i \times \{i\}$ of $\Gamma_0 + \Gamma_1$ $(i = 0,1)$.

(2) If Γ is a graph and $\xi \notin \Gamma$ then $\Gamma + \xi = \Gamma \cup \{\xi\}$. For $x,y \in \Gamma + \xi$ we set x—y if and only if $x = \xi$ or $y = \xi$ or $x,y \in \Gamma$ and x—y in Γ.

(3) By a representation of a graph Γ we mean an injective function $\varphi: \Gamma \to \mathcal{F}$ such that for any $x, y \in \Gamma$, x—y in Γ if and only if $\varphi(x)$—$\varphi(y)$ in $\mathcal{F}$.

Consider the following statement.

(*) Every finite graph Γ has a representation.

We will show that (*) is false.

Definition. Let $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{A} \in \mathcal{D}_1$ if and only if $u,v \in \mathcal{A}$, $u \mid v$ implies $u = v$. $\mathcal{B} \subseteq \mathcal{A} \in \mathcal{D}_1$ implies $\mathcal{B} \in \mathcal{D}_1$.

Lemma 4.1. *Assume* (*). *Let* Γ *be a finite graph. Then there exists a representation* φ *of* Γ *such that* $\varphi(\Gamma) \in \mathcal{D}_1$.

Proof. Suppose there exists no such representation of Γ. Let $n = |\Gamma|^2 + 1$, $T = \{1, ..., n\}$. Set $\Gamma_1 = \Gamma \times T$. For $(x,i), (y,j) \in \Gamma \times T$ we define (x,i)—(y,j) if and only if $i \neq j$ and $x \neq y$, or $i = j$ and x—y. Let $\theta : \Gamma_1 \to \mathcal{F}$ be a representation. Let $i \in T$ and $\psi : \Gamma \to \mathcal{F}$ be the representation $\psi(x) = \theta(x,i)$. By assumption there must exist $x,y \in \Gamma$ such that $x \neq y$ and $\theta(x,i) \mid \theta(y,i)$. Set $\delta(i) = (x,y)$. Hence we have the function $\delta : T \to \Gamma \times \Gamma$. Since $|T| > |\Gamma \times \Gamma|$ there must exist $i,j \in T$, $i \neq j$ such that $\delta(i) = \delta(j) = (x,y)$. Hence $x \neq y$ and $\theta(x,i) \mid \theta(y,i)$, $\theta(x,j) \mid \theta(y,j)$. Since $\theta(y,i)$—$\theta(x,j)$ we get $\theta(x,i) \to \theta(x,j)$. Since $\theta(y,j)$—$\theta(x,i)$ we get $\theta(x,j) \to \theta(x,i)$.

So $\theta(x,i)$—$\theta(x,j)$ implying (x,i)—(x,j), a contradiction.

Definition. Let $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{A} \in \mathcal{D}_2$ if and only if $\mathcal{A} \in \mathcal{D}_1$ and for all $u, v \in \mathcal{A}$, $u \sim v$ implies $u = v$. $\mathcal{B} \subseteq \mathcal{A} \in \mathcal{D}_2$ implies $\mathcal{B} \in \mathcal{D}_2$.

Lemma 4.2. *Assume* $(*)$ *and let* Γ *be a finite graph. Then there exists a representation* ψ *of* Γ *such that* $\psi(\Gamma) \in \mathcal{D}_2$.

Proof. Suppose there exists no such representation of Γ. Let Γ_1 and T be as in the proof of Lemma 4.1. By Lemma 4.1 there exists a representation χ of Γ_1 such that $\chi(\Gamma_1) \in \mathcal{D}_1$. Let $i \in T$ and $\theta : \Gamma \to \mathcal{F}$ be the representation $\theta(x) = \chi(x,i)$. Then $\theta(\Gamma) \in \mathcal{D}_1$ and hence by our assumption there exist $x, y \in \Gamma$ such that $x \neq y$ and $\chi(x,i) \sim \chi(y,i)$. Set $\rho(i) = (x,y)$. We have $\rho : T \to \Gamma \times \Gamma$. Since $|T| > |\Gamma \times \Gamma|$ there exist $i, j \in T$, $i \neq j$ such that $\rho(i) = \rho(j) = (x,y)$. Hence $x \neq y$ and $\chi(x,i) \sim \chi(y,i)$ and $\chi(x,j) \sim \chi(y,j)$. Since $\chi(x,i)$—$\chi(y,j)$ and $\chi(x,j)$—$\chi(y,i)$ we get $\chi(x,i) \to \chi(x,j)$ and $\chi(x,j) \to \chi(x,i)$. Hence $\chi(x,j)$—$\chi(x,i)$ implying (x,i)—(x,j), a contradiction.

Definition. Let $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{A} \in \mathcal{D}_3$ if and only if $\mathcal{A} \in \mathcal{D}_2$ and each word in $\mathcal{A}$ is primitive. $\mathcal{B} \subseteq \mathcal{A} \in \mathcal{D}_3$ implies $\mathcal{B} \in \mathcal{D}_3$.

Lemma 4.3. *Assume* $(*)$ *and let* Γ *be a finite graph. Then there exists a representation* φ *of* Γ *such that* $\varphi(\Gamma) \in \mathcal{D}_3$.

Proof. Assume Γ has no such representation and let $\Gamma_1 = \Gamma + \Gamma$. By Lemma 4.2 there exists a representation φ of Γ_1 such that $\varphi(\Gamma_1) \in \mathcal{D}_2$. For each $i = 0, 1$, $\theta_i : \Gamma \to \mathcal{F}$ given by $\theta_i(x) = \varphi(x,i)$ is a representation of Γ and $\theta_i(\Gamma) \in \mathcal{D}_2$. By assumption there exists $x, y \in \Gamma$ such that $\theta_0(x) = \varphi(x,0)$ and $\theta_1(y) = \varphi(y,1)$ are both not primitive. Since $(x,0)$—$(y,1)$ we get $\varphi(x,0)$—$\varphi(y,1)$ and so by Lemma 1.4(vii) $\varphi(x,0) \sim \varphi(y,1)$. Since $\varphi(\Gamma_1) \in \mathcal{D}_2$ we get $\varphi(x,0) = \varphi(y,1)$ implying $(x,0) = (y,1)$ a contradiction.

Definition. Let $\mathcal{A} \subseteq \mathcal{F}$. Then $\mathcal{A} \in \mathcal{D}$ if and only if $\mathcal{A} \in \mathcal{D}_3$ and for all $\omega_1, \omega_2 \in \mathcal{A}$ $|\omega_1| < 2|\omega_2|$. $\mathcal{B} \subseteq \mathcal{A} \in \mathcal{D}$ implies $\mathcal{B} \in \mathcal{D}$.

Remark 4.4. Thus for $\mathcal{A} \subseteq \mathcal{F}$, $\mathcal{A} \in \mathcal{D}$ if and only if for all $\omega_1, \omega_2 \in \mathcal{A}$;

(i) $|\omega_1| < 2|\omega_2|$
(ii) $\omega_1 \sim \omega_2$ implies $\omega_1 = \omega_2$.
(iii) ω_1 is primitive.
(iv) $\omega_1 | \omega_2$ implies $\omega_1 = \omega_2$.
(v) $\omega_1 \to \omega_2$ implies $\omega_1 | \omega_2^2$ (this is by Lemma 1.3(v)).

Lemma 4.5. *Assume* $(*)$ *and let* Γ *be a finite graph. Then there exists a representa-*

tion φ of Γ such that $\varphi(\Gamma) \in \mathcal{D}$.

Proof. Let $\Gamma_1 = \Gamma + \Gamma$. By Lemma 4.3 there exists a representation θ of Γ_1 such that $\theta(\Gamma_1) \in \mathcal{D}_3$. Choose $\alpha \in \Gamma_1$ such that $|\theta(\alpha)|$ is maximal. Then $\alpha = (z, i)$ for some $z \in \Gamma$ and $i \in \{0, 1\}$. Let $j = 1 - i$ and $u = \theta(\alpha)$. Then $\varphi : \Gamma \to \mathcal{F}$ given by $\varphi(x) = \theta(x, j)$ is a representation of Γ and $\varphi(\Gamma) \in \mathcal{D}_3$. Moreover for any $v \in \varphi(\Gamma)$, $|v| \leqslant |u|$ and v—u, $v \nmid u$. By Lemma 1.3(v) $|u| < 2|v|$. So for any $v, v' \in \varphi(\Gamma)$, $|v'| \leqslant |u| < 2|v|$, proving that $\varphi(\Gamma) \in \mathcal{D}$.

Lemma 4.6. *Let P be a non-empty finite set. Then there exist a non-empty finite set Ω and $\theta : \Omega \times \Omega \to P$ such that $\theta(\alpha, \beta) = \theta(\beta, \alpha)$ for all $\alpha, \beta \in \Omega$ and so that for any $\delta : \Omega \to P$ there exist $\alpha, \beta \in \Omega$, $\alpha \neq \beta$ with $\theta(\alpha, \beta) = \delta(\alpha) = \delta(\beta)$.*

Proof. We prove the lemma by induction on $|P|$. If $|P| = 1$ we can take Ω to be any two element set and θ the only possible function. So let $|P| > 1$. By induction hypothesis, for each $t \in P$, there exists a finite non-empty set Ω_t, $\theta_t : \Omega_t \times \Omega_t \to P \setminus \{t\}$ such that $\theta_t(\alpha, \beta) = \theta_t(\beta, \alpha)$ for all $\alpha, \beta \in \Omega_t$ and so that for any $\delta : \Omega_t \to P \setminus \{t\}$ there exist $\alpha, \beta \in \Omega_t$, $\alpha \neq \beta$ with $\delta(\alpha) = \delta(\beta) = \theta_t(\alpha, \beta)$. We can assume without loss of generality that the Ω_t's are disjoint. Let $\Omega = \bigcup_{t \in P} \Omega_t \cup \{\xi\}$ where $\xi \notin \Omega_t$ for any $t \in P$. Define $\theta : \Omega \times \Omega \to P$ as follows: $\theta(\alpha, \xi) = \theta(\xi, \alpha) = t$ for all $\alpha \in \Omega_t$, $\theta = \theta_t$ on $\Omega_t \times \Omega_t$, θ sends rest of $\Omega \times \Omega$ to any single point in P. Then it is obvious that $\theta(\alpha, \beta) = \theta(\beta, \alpha)$ for all $\alpha, \beta \in P$. Now let $\delta : \Omega \to P$ and let $t = \delta(\xi)$. If for any $\alpha \in \Omega_t$, $\delta(\alpha) = t$, then $t = \delta(\alpha) = \delta(\xi) = \theta(\alpha, \xi)$. Otherwise $\delta' : \Omega_t \to P \setminus \{t\}$ where δ' is the restriction of δ to Ω_t. But now of course, the result follows by our construction of Ω_t and θ_t above.

Construction 4.7. Let $l \in \mathbf{Z}^+$. We construct a finite graph Λ_l as follows. Let $J = \{1, 2, ..., 2l^2 + 1\}$, $m = \binom{2l^2+1}{l^2}$, $P = \{1, ..., m\}$. Then J has exactly m distinct subsets with l^2 elements, say J_i $(i \in P)$. Next let Ω, θ be as in Lemma 4.6 associated with P. Let $\Lambda_l = J \times \Omega$. If $\alpha \in \Omega$ we set $J^{(\alpha)} = J \times \{\alpha\}$ and $J_i^{(\alpha)} = J_i \times \{\alpha\}$ $(i = 1, ..., m)$. Λ_l is the disjoint union of the $J^{(\alpha)}$'s. If $x \in J^{(\alpha)}$ and $y \in J^{(\beta)}$, we define x—y if and only if $\alpha = \beta$, or $\alpha \neq \beta$ and $x \notin J^{(\alpha)}_{\theta(\alpha,\beta)}$ or $y \notin J^{(\beta)}_{\theta(\alpha,\beta)}$. In other words, $x \not\!\!-\, y$ if and only if $x \in J^{(\alpha)}_{\theta(\alpha,\beta)}$, $y \in J^{(\beta)}_{\theta(\alpha,\beta)}$ and $\alpha \neq \beta$. — is evidently reflexive; it is symmetric because $\theta(\alpha, \beta) = \theta(\beta, \alpha)$ for all $\alpha, \beta \in \Omega$. For $l \in \mathbf{Z}^+$ the symbol Λ_l will be reserved for the above graph for the rest of this section (all other symbols may be used for other things).

Lemma 4.8. *Let $l \in \mathbf{Z}^+$ and let $\leqslant$ be any linear order on Λ_l. Then there exist distinct elements $x_1, ..., x_{2l^2+1}, y_1, ..., y_{2l^2+1} \in \Lambda_l$ such that $x_1 < x_2 < ... < x_{2l^2+1}$, $y_1 < y_2 < ... < y_{2l^2+1}$ and for $i, j \in \{1, ..., 2l^2+1\}$, x_i—y_j if and only if i is odd or j is odd.*

Proof. We follow the notation of Construction 4.7. Let $\delta : \Omega \to P$ be defined as follows.

For $\alpha \in \Omega$ write the $2l^2+1$ element of $J^{(\alpha)}$ as $a_1 < a_2 < ... < a_{2l^2+1}$. Let $A = \{a_2, a_4, ..., a_{2l^2}\}$. Then $|A| = l^2$ and therefore $A = J_i^{(\alpha)}$ for some $i \in P$. We set $\delta(\alpha) = i$. By Lemma 4.6 there exist $\alpha, \beta \in \Omega$, $\alpha \neq \beta$ with $\delta(\alpha) = \delta(\beta) = \theta(\alpha,\beta) = k \in P$. Write elements of $J^{(\alpha)}$ as $x_1 < x_2 < ... < x_{2l^2+1}$ and elements of $J^{(\beta)}$ as $y_1 < y_2 < ... < y_{2l^2+1}$. Then $J_k^{(\alpha)} = \{x_2, ..., x_{2l^2}\}$ and $J_k^{(\beta)} = \{y_2, ..., y_{2l^2}\}$. Since $k = \theta(\alpha,\beta)$ we see by definition of our graph Λ_l that for $i,j \in \{1,2,...,2l^2+1\}$, x_i—y_j if and only if $x_i \notin J_k^{(\alpha)}$ or $y_j \notin J_k^{(\beta)}$. This proves the lemma.

Lemma 4.9. *Let $l \in \mathbf{Z}^+$ and $\mathcal{A}$, $\mathcal{B}$ two disjoint subsets of $\mathcal{F}$ each with l^2 elements. Suppose further that u-+-v for any $u \in \mathcal{A}$, $v \in \mathcal{B}$. Then either there exists a subset $\mathcal{M}$ of $\mathcal{A}$, $v \in \mathcal{B}$ such that $|\mathcal{M}| = l$ and $u \nrightarrow v$ for any $u \in \mathcal{M}$, or else there exists a subset $\mathcal{N}$ of $\mathcal{B}$, $u \in \mathcal{A}$ such that $|\mathcal{N}| = l$ and $v \nrightarrow u$ for any $v \in \mathcal{N}$.*

Proof. Suppose there exists no $\mathcal{N} \subseteq \mathcal{B}$, $u \in \mathcal{A}$ with the prescribed property. Let $\mathcal{M}$ be any subset of $\mathcal{A}$ such that $|\mathcal{M}| = l$. For $u \in \mathcal{M}$, let $\mathcal{Y}_u = \{v \mid v \in \mathcal{B} \text{ and } v \nrightarrow u\}$. By our assumption $|\mathcal{Y}_u| < l$. So $|\bigcup_{u\in\mathcal{M}} \mathcal{Y}_u| < l^2 = |\mathcal{B}|$. Hence there exists $v \in \mathcal{B}$ such that $v \notin \mathcal{Y}_u$ for any $u \in \mathcal{M}$. Thus $v \to u$ for any $u \in \mathcal{M}$. Since v-+-u it must be that $u \nrightarrow v$ for all $u \in \mathcal{M}$. This proves the lemma.

Lemma 4.10. *Let $l \in \mathbf{Z}^+$, φ a representation of Λ_l and $\leqslant$ any linear order on $\varphi(\Lambda_l)$. Then there exist distinct elements $u_1, ..., u_{2l+1}, v \in \varphi(\Lambda_l)$ with $u_1 < u_2 < ... < u_{2l+1}$ such that u_i—v for i odd and $u_i \nrightarrow v$ for i even, $1 \leqslant i \leqslant 2l+1$.*

Proof. By Lemma 4.8 there exist distinct words $u_1, ..., u_{2l^2+1}, v_1, ..., v_{2l^2+1} \in \varphi(\Lambda_l)$ such that $u_1 < u_2 < ... < u_{2l^2+1}$, $v_1 < v_2 < ... < v_{2l^2+1}$; and u_i-+-v_j if and only if i,j are both even. Let $\mathcal{A} = \{u_2, ..., u_{2l^2}\}$, $\mathcal{B} = \{v_2, ..., v_{2l^2}\}$. By Lemma 4.9 and symmetry we may assume that there exists a subset $\mathcal{M}$ of $\mathcal{A}$, $v_j \in \mathcal{B}$ such that $|\mathcal{M}| = l$ and $u_k \nrightarrow v_j$ for any $u_k \in \mathcal{M}$. Write $\mathcal{M} = \{u_{i_1} < ... < u_{i_l}\}$ and let $\mathcal{M}_1 = \{u_{i_1-1}, u_{i_2-1}, ..., u_{i_l-1}, u_{i_l+1}\}$. $u_k \in \mathcal{M}_1$ implies k is odd whence u_k—v_j. Furthermore

$$u_{i_1-1} < u_{i_1} < u_{i_2-1} < u_{i_2} < ... < u_{i_l-1} < u_{i_l} < u_{i_l+1}\,.$$

Setting $v = v_j$ and relabeling the u's we have the result.

Definition. Let $V \in \mathcal{F}$, V primitive, $t \in \mathbf{Z}^+$. Then $\omega \in (V,t)$ if and only if $\omega \in \mathcal{F}$, ω is primitive, $\omega \equiv xV^i$ for some $x \in \mathcal{F}^1$, $i \in \mathbf{Z}^+$ such that $i \geqslant t$, $|x| \leqslant |\omega|/_t$, $|V| \leqslant |\omega|/_t$. Note that $(V,t) \supseteq (V,t+1)$. Also $(\mathcal{V},t) = \{\omega \mid \omega \in \mathcal{F} \text{ and } \omega \in (V,t)$ for some primitive $V \in \mathcal{F}\}$.

The following is the crucial step.

Lemma 4.11. *Let $t \in \mathbf{Z}^+$ and set $l = 64t^2$. If φ is a representation of $\Lambda_l + \Lambda_l$ such that $\varphi(\Lambda_l + \Lambda_l) \in \mathcal{D}$ then $\varphi(\Lambda_l + \Lambda_l) \cap (\mathcal{V}, 2t) \neq \emptyset$.*

Proof. Choose $\omega \in \varphi(\Lambda_l + \Lambda_l)$ with $|\omega|$ minimal. It suffices to prove that $\omega \in (\mathcal{V}, 2t)$. Now $\varphi(\gamma, \xi) = \omega$ for some $\gamma \in \Lambda_l$, $\xi \in \{0,1\}$. Consider the representation ψ of Λ_l given by $\psi(\beta) = \varphi(\beta, 1-\xi)$. Let $\mathcal{A} = \psi(\Lambda_l)$. Then $\mathcal{A} \in \mathcal{D}$. For any $u \in \mathcal{A}$ we have u—ω. Let $n = |\omega|$. Then since $\varphi(\Lambda_l + \Lambda_l) \in \mathcal{D}$ we have,

$$(12) \qquad \begin{aligned} &|\omega| = n, \quad \omega \text{ is primitive;} \\ &u \nmid \omega, \quad u \nmid \omega^2, \quad n \leqslant |u| < 2n \quad \text{for all} \quad u \in \mathcal{A}\,. \end{aligned}$$

Thus if $u \in \mathcal{A}$ then $\omega^2 = aub$ for some $a,b \in \mathcal{F}^1$. As $u \nmid \omega$ we must have $|a|, |b| < |\omega|$. Hence for some $c,d \in \mathcal{F}$, $\omega = ac = db$ implying $u = cd$. Thus

$$(13) \qquad \omega = ac = db, \ u = cd \quad \text{for some} \quad a,b,c,d \in \mathcal{F}^1.$$

Fix one such decomposition for each $u \in \mathcal{A}$. Let $u_1, u_2 \in \mathcal{A}$ and corresponding to (13) write the decompositions of u_1, u_2 as follows:

$$(14) \qquad \begin{aligned} &\omega = a_1c_1 = d_1b_1 = a_2c_2 = d_2b_2\,, \quad u_1 = c_1d_1\,, \\ &u_2 = c_2d_2, \quad a_1,b_1,a_2,b_2,c_1,c_2,d_1,d_2 \in \mathcal{F}^1. \end{aligned}$$

Define $u_1 \leqslant u_2$ if and only if $|a_1| \geqslant |a_2|$. Evidently, $\leqslant$ is transitive and $u_1 \leqslant u_2$ or $u_2 \leqslant u_1$. If $u_1 \leqslant u_2 \leqslant u_1$ then $|a_1| = |a_2|$ and by (14) $a_1 = a_2$ and $c_1 = c_2$. Since $d_1b_1 = d_2b_2$ we have $d_1 |_i d_2$ or $d_2 |_i d_1$. If $d_1 |_i d_2$ then $u_1 = c_1d_1 = c_2d_1 |_i c_2d_2 = u_2$. So $u_1 | u_2$ and since $\mathcal{A} \in \mathcal{D}$ we get $u_1 = u_2$. Similarly $d_2 |_i d_1$ implies $u_1 = u_2$. It follows that $\leqslant$ is a linear order on $\mathcal{A}$. Next assume $u_1 < u_2$. Then $|a_1| > |a_2|$ and by (14), $|c_1| < |c_2|$ and $c_1 |_f c_2$. Since $d_1b_1 = d_2b_2$ we have $d_1 |_i d_2$ or $d_2 |_i d_1$. If $d_1 |_i d_2$ then by Lemma 1.2 (i), $u_1 = c_1d_1 | c_2d_2 = u_2$, a contradiction since $u_1 \neq u_2$. So $d_2 |_i d_1$. Thus we have

$$(15) \qquad c_1 |_f c_2 \quad \text{and} \quad d_2 |_i d_1 \qquad (\text{if } u_1 < u_2)\,.$$

Continuing with the same u_1, u_2 assume there exists $u_3 \in \mathcal{A}$ such that $u_1 < u_2 < u_3$. Correspondingly let

$$(16) \qquad u_3 = c_3d_3, \quad \omega = a_3c_3 = d_3b_3\,; \quad a_3,b_3,c_3,d_3 \in \mathcal{F}^1\,.$$

By (15) we get

$$(17) \qquad c_1 |_f c_2 |_f c_3\,; \quad d_3 |_i d_2 |_i d_1.$$

By Lemma 1.2 (i) we get

$$(18) \qquad u_2 = c_2d_2 \,|\, c_3d_1\,.$$

(18) holds whenever $u_1 < u_2 < u_3$ and the corresponding decompositions are as in (14) and (16).

Now, as $\leqslant$ is a linear order on $\mathcal{A} = \psi(\Lambda_l)$, we can apply Lemma 4.10 to obtain distinct words $u_1, \ldots, u_{2l+1}$, $v \in \mathcal{A}$ such that

(19) $\quad u_1 < u_2 < \ldots < u_{2l+1}$; $u_i \text{---} v$ for i odd;

$u_i \not\rightarrow v$ for i even where $1 \leq i \leq 2l+1$.

Corresponding to (13) write

(20) $\quad u_i = c_i d_i, \quad \omega = a_i c_i = d_i b_i, \quad a_i, b_i, c_i, d_i \in \mathcal{F}^1, \quad 1 \leq i \leq 2l+1$.

Let $T = \{1,3,\ldots,2l-1,2l+1\}$. For each $i \in T$, $u_i \rightarrow v$. Since $\mathcal{A} \in \mathcal{D}$ we get $u_i \mid v^2$. Thus by (12), (20) and Lemma 1.3 (ii),

(21) $\quad v^2 = g_i c_i d_i h_i, \quad g_i, h_i \in \mathcal{F}^1; \quad |g_i| \leq |v| < 2n$ for all $i \in T$.

Next consider the map $\mu: T \rightarrow [0,n]$ given by $\mu(i) = |c_i|$. Now $|T| = l+1 = 64t^2+1$. Also $T = \bigcup_{\alpha=1}^{8t} \mu^{-1}[(\alpha-1)n/8t, \alpha n/8t]$. It cannot be that the cardinality of each $\mu^{-1}[(\alpha-1)n/8t, \alpha n/8t]$ is $\leq 8t$, lest $|T| \leq 64t^2$. We see therefore that there exists a subset T_1 of T such that $|T_1| = 8t+1$ and for all $i,j \in T_1$, $|\mu(i) - \mu(j)| \leq n/8t$. Thus

(22) $\quad ||c_i| - |c_j|| \leq n/8t \quad$ for all $\quad i,j \in T_1; \quad T_1 \subseteq T; \quad |T_1| = 8t+1$.

Next consider the map $\nu: T_1 \rightarrow [0,2n]$ given by $\nu(i) = |g_i|$. Then $T_1 = \bigcup_{\alpha=1}^{8t} \nu^{-1}[(\alpha-1)n/4t, \alpha n/4t]$. It cannot be that each $\nu^{-1}[(\alpha-1)n/4t, \alpha n/4t]$ has at most one element lest $|T_1| \leq 8t$ contradicting (22). It follows that there exist $i,j \in T_1$ such that $i < j$ and $|\nu(i) - \nu(j)| \leq n/4t$. Then

(23) $\quad ||g_i| - |g_j|| \leq n/4t \quad$ for some $i,j \in T_1$, $i < j$.

By (21) we get

(24) $\quad g_i c_i d_i h_i = g_j c_j d_j h_j = v^2$.

By (12), (20), $|c_j d_j| = |u_j| \geq |\omega| = |a_j c_j|$. Hence $|d_j| \geq |a_j|$. By (20), we have $\omega = a_j c_j = d_j b_j$. Thus $d_j = a_j p$ for some $p \in \mathcal{F}^1$. Let $\omega_1 = c_j a_j \equiv \omega$. Then $c_j d_j = \omega_1 p$. By (15) $c_i \mid_f c_j$, $d_j \mid_i d_i$. So for some $q, q_1 \in \mathcal{F}^1$, $c_j = q c_i$ and $d_i = d_j q_1$. Let $\omega_2 = c_i a_j$. Thus by (22), (24),

(25) $\quad g_i \omega_2 p q_1 h_i = g_j q \omega_2 p h_j; \quad q\omega_2 = \omega_1 \equiv \omega$;

$|q| = ||c_i| - |c_j|| \leq n/8t$.

Next we claim that $g_i \neq g_j q$. For suppose $g_i = g_j q$. Then by (24), $v^2 = g_i c_i d_i h_i = g_j q c_i d_i h_i = g_j c_j d_i h_i$ (since $c_j = q c_i$). Hence $c_j d_i \mid v^2$. Now since i and j are odd, $k = i+1$ is even and $i < k < j$. By (18), (19) and (20) we get $u_k = c_k d_k \mid c_j d_i \mid v^2$ contradicting (19). This contradiction shows that $g_i \neq g_j q$. Thus in (25), putting $A_1 = g_i$, $A_2 = g_j q$, $B_1 = p q_1 h_i$, $B_2 = p h_j$ we get by (23),

(26) $\quad A_1 \omega_2 B_1 = A_2 \omega_2 B_2; \quad A_1 \neq A_2$;

$$||A_1| - |A_2|| \leq \frac{n}{4t} + \frac{n}{8t} = \frac{3n}{8t} .$$

By Lemma 1.2 (v) there exist $x, y \in \mathcal{F}^1$, $m \in \mathbf{N}$ such that $\omega_2 = x(yx)^m$, $|yx| = ||A_1| - |A_2||$. Let $U = yx$, $z = qx$. Then by (26)

$$(27) \qquad 0 < |U| \leqslant \frac{3n}{8t} < \frac{n}{2t}.$$

Also, by (25)

$$(28) \qquad \omega \equiv \omega_1 = q\omega_2 = (qx)(yx)^m = zU^m.$$

By (25) and (27),

$$(29) \qquad |z| = |q| + |x| \leqslant |q| + |U| \leqslant \frac{n}{8t} + \frac{3n}{8t} = \frac{n}{2t}.$$

Hence by (12), (27), (28), (29) and the fact that $2t - 1 \geqslant 1$ we get

$$m|U| = |U^m| = n - |z| \geqslant n - \frac{n}{2t} = (2t-1)\frac{n}{2t} > |U|(2t-1).$$

Since $|U| > 0$ we get $m > 2t - 1$ whence

$$(30) \qquad m \geqslant 2t.$$

Now let $U = V^\lambda$ where $\lambda \in \mathbf{Z}^+$ and V primitive. Hence by (27), (28), (29), (30) we get

$$(31) \qquad |V| \leqslant |U| \leqslant \frac{n}{2t}; \quad \omega \equiv zV^{\lambda m}; \quad |z| \leqslant \frac{n}{2t}; \quad \lambda m \geqslant m \geqslant 2t.$$

It follows by (12) that $\omega \in (V, 2t) \subseteq (\mathcal{V}, 2t)$. This proves the lemma.

Lemma 4.12. *Let $U, V \in \mathcal{F}$ be primitive words, $t \in \mathbf{Z}^+$, $t \geqslant 7$. Let $\omega_1 \in (U, 2t)$, $\omega_2 \in (V, 2t)$, $|\omega_1| = n$, $|\omega_2| = m$. Suppose further that $m < 2n$, $n < 2m$ and that $\omega_1 — \omega_2$. Then $U \equiv V$, $\omega_1 \in (V, t)$ and $\omega_2 \in (U, t)$.*

Proof. By symmetry let $|U| \geqslant |V|$. Now for some $x, y \in \mathcal{F}^1$, $j, k \in \mathbf{Z}^+$,

$$\begin{aligned} &\omega_1, \omega_2 \text{ are primitive;} \\ (32) \qquad &\omega_1 \equiv xU^k,\ |x| \leqslant n/2t,\ |U| \leqslant n/2t,\ k \geqslant 2t; \\ &\omega_2 \equiv yV^j,\ |y| \leqslant m/2t,\ |V| \leqslant m/2t,\ j \geqslant 2t. \end{aligned}$$

Let $i = \frac{1}{2}k$ if k is even and $\frac{1}{2}(k-1)$ if k is odd. Then $k - 1 \leqslant 2i \leqslant k$. Hence $U^{2i} | xU^k \to \omega_1$ implying $U^{2i} \to \omega_1$. Also $|\omega_1| = |xU^k| \geqslant |U^{2i}|$. By Lemma 1.3 (iv), $U^i | \omega_1$. Since $k \geqslant 2t$ (so if k is odd then $k \geqslant 2t + 1$) we get $i \geqslant t \geqslant 7$. Also,

$$|U^i| = i|U| \geqslant \frac{k-1}{2}|U| = \tfrac{1}{2}(|U^k| - |U|) = \tfrac{1}{2}(|\omega_1| - |x| - |U|)$$

$$\geqslant \frac{1}{2}\left(n - \frac{n}{2t} - \frac{n}{2t}\right) = \frac{n}{2}\left(\frac{t-1}{t}\right) > \frac{m}{4}\left(\frac{t-1}{t}\right) \geqslant \frac{3m}{2t}.$$

Since $i-4>\frac{1}{3}i$ we get

(33) $$|U^{i-4}|>\tfrac{1}{3}|U^i|>\frac{m}{2t}\geqslant|y|.$$

Moreover

(34) $$|U^3|\leqslant\frac{3n}{2t}<\frac{3m}{t}<m-\frac{m}{2t}\leqslant|\omega_2|-|y|=|V^j|.$$

Next $U^i|\omega_1\to\omega_2\equiv V^jy$ where by $U^i\to V^jy$. So by Lemma 1.3(ii),

(35) $$SU^iT=(V^jy)^r \quad \text{for some } r\in\mathbb{Z}^+,\ r\geqslant 2,\ S,T\in\mathcal{F}^1,\ |S|<|V^jy|.$$

We claim that $U^2|V^j$. If $|SU^2|\leqslant|V^j|$ then $SU^2\,|_{\mathrm{i}}\,V^j$ and $U^2|V^j$. Next assume $|SU^2|>|V^j|$. Then by (33), (35) we get $|S|<|V^jy|<|SU^{i-2}|$. Then by (35), $S\,|_{\mathrm{i}}\,V^jy\,|_{\mathrm{i}}\,SU^{i-2}$ and by Lemma 1.2 (ii),

(36) $$V^jy=SU^\alpha D,\quad |D|<|U|,\quad U=DF \quad \text{for some } D,F\in\mathcal{F}^1,\quad \alpha\in\mathbb{N}.$$

Now $|SU^\alpha|\leqslant|V^jy|<|SU^{i-2}|$ implying $\alpha\leqslant i-3$. By (35), $SU^iT=(SU^\alpha D)(V^jy)^{r-1}$ So $SU^\alpha DFU^{i-\alpha-1}T=(SU^\alpha D)(V^jy)^{r-1}$. Therefore

(37) $$FU^{i-\alpha-1}T=(V^jy)^{r-1},\quad i-\alpha-1\geqslant 2.$$

By (34), (36),

$$|FU^2|\leqslant|U^3|<|V^j|.$$

So by (37), $FU^2\,|_{\mathrm{i}}\,V^j$ implying $U^2|V^j$. By Lemma 1.4 (iv), $U\equiv V$. Hence for some $A,B\in\mathcal{F}^1$, $U=AB$, $V=BA$. So

$$\omega_1\equiv x(AB)^k\equiv(BxA)(BA)^{k-1}=x_1V^{k-1} \quad \text{where} \quad x_1=BxA.$$

$$\omega_2\equiv y(BA)^j\equiv(AyB)(AB)^{j-1}=y_1U^{j-1} \quad \text{where} \quad y_1=AyB.$$

Also,

$$|x_1|=|x|+|U|\leqslant\frac{n}{2t}+\frac{n}{2t}=\frac{n}{t};$$

$$|y_1|=|y|+|V|\leqslant\frac{m}{2t}+\frac{m}{2t}=\frac{m}{t};$$

$$|U|=|V|\leqslant\frac{m}{2t}<\frac{m}{t};\qquad |V|=|U|\leqslant\frac{n}{2t}<\frac{n}{t};$$

$$k-1\geqslant 2t-1\geqslant t;\qquad j-1\geqslant 2t-1\geqslant t.$$

Hence $\omega_2\in(U,t)$ and $\omega_1\in(V,t)$. This proves the lemma.

Lemma 4.13. *Assume (*). Let* $(\Gamma,—)$ *be a finite graph and* $t\in\mathbb{Z}^+$, $t\geqslant 7$. *Then there exist a representation* φ *of* Γ *and a primitive word* $V\in\mathcal{F}$ *such that* $\varphi(\Gamma)\in\mathcal{D}$ *and* $\varphi(\Gamma)\subseteq(V,t)$.

Proof. Let $\xi \notin \Gamma$ and let $\Gamma' = \Gamma + \xi$. Let $l = 64t^2$ and set $\Lambda = \Lambda_l + \Lambda_l$. Set $\Gamma_1 = \Gamma' \times \Lambda$ where for $(x,y), (x',y') \in \Gamma_1$ we define $(x,y) \text{---} (x',y')$ if and only if $x = x'$ and $y \text{---} y'$, or $x \neq x'$ and $x \text{---} x'$. By Lemma 4.5 there exists a representation ψ of Γ_1 such that $\psi(\Gamma_1) \in \mathcal{D}$. Let $x \in \Gamma'$. Consider the representation θ of Λ given by $\theta(y) = \psi(x,y)$. Then $\theta(\Lambda) \in \mathcal{D}$. By Lemma 4.11, $\theta(\Lambda) \cap (\mathcal{V}, 2t) \neq \emptyset$. So for each $x \in \Gamma'$ there exists $y_x \in \Lambda$ such that $\psi(x, y_x) \in (\mathcal{V}, 2t)$. Consider the representation φ of Γ' given by $\varphi(x) = \psi(x, y_x)$. Then $\varphi(\Gamma') \in \mathcal{D}$ and $\varphi(\Gamma') \subseteq (\mathcal{V}, 2t)$. Let $v = \varphi(\xi)$. So $v \in (V, 2t)$ for some primitive $V \in \mathcal{F}$. Let $x \in \Gamma$, $\varphi(x) = u$. Since $x \text{---} \xi$ we get $u \text{---} v$. Since $u \in (\mathcal{V}, 2t)$ we have $u \in (U, 2t)$ for some primitive U in $\mathcal{F}$. Since $\varphi(\Gamma') \in \mathcal{D}$, by Lemma 4.12, $u \in (V, t)$. It follows that $\varphi(\Gamma) \subseteq (V, t)$. We see that φ restricted to Γ is the required representation.

Definition. Let $V \in \mathcal{F}$ be primitive.

(1) $\omega \in (V,0,1)$ if and only if $\omega \in \mathcal{F}$, ω is primitive, $\omega = aV^iyV^jb$ for some $a,b \in \mathcal{F}^1$, $y \in \mathcal{F}$, $i,j \in \mathbf{Z}^+$, $i,j \geq 3$ such that $ba = V$, $|V^2yV^2| \leq \frac{1}{4}|\omega|$, $|\omega| \leq 2|V^{i+j+1}|$, $V \nmid_{\mathrm{i}} y$, $V \nmid_{\mathrm{f}} y$.

(2) $\omega \in (V,0,2)$ if and only if $\omega \in \mathcal{F}$, ω is primitive, $\omega = x_1V^ix_2$ for some $x_1, x_2 \in \mathcal{F}^1$, $i \in \mathbf{Z}^+$, $i \geq 3$ such that $V \nmid_{\mathrm{i}} x_2$, $V \nmid_{\mathrm{f}} x_1$, $V^5 \nmid_{\mathrm{i}} x_2x_1V^i$, $V^5 \nmid_{\mathrm{f}} V^ix_2x_1$, $|x_2x_1| \leq |V^{i-2}|$.

Lemma 4.14. *Let $V \in \mathcal{F}$ be primitive. Then $(V,20) \subseteq (V,0,1) \cup (V,0,2)$.*

Proof. Let $\omega \in (V,20)$. Then $\omega \in \mathcal{F}$ and

$$(38) \qquad |\omega| = n \in \mathbf{Z}^+,\ \omega \text{ is primitive};\ \omega \equiv x'V^k \text{ for some } x' \in \mathcal{F}^1,$$
$$k \in \mathbf{Z}^+,\ k \geq 20;\ |x'|, |V| \leq n/20 .$$

Now $x' = V^{\epsilon_1}xV^{\epsilon_2}$ for some $\epsilon_1, \epsilon_2 \in \mathbf{N}$, $x \in \mathcal{F}^1$, $V \nmid_{\mathrm{i}} x$, $V \nmid_{\mathrm{f}} x$. Let $l = k + \epsilon_1 + \epsilon_2$. Then $\omega \equiv xV^l$, $l \geq k$. Now if $x = 1$ then $\omega \equiv V^l$ and we have a contradiction to Lemma 1.4 (ii). Thus by (38) we have

$$(39) \qquad |\omega| = n \in \mathbf{Z}^+;\ \omega \text{ is primitive};\ \omega \equiv xV^l \text{ for some } x \in \mathcal{F},$$
$$l \in \mathbf{Z}^+,\ l \geq 20;\ |x|, |V| \leq n/20;\ V \nmid_{\mathrm{i}} x,\ V \nmid_{\mathrm{f}} x .$$

Hence

$$|x| \leq \frac{n}{20} \leq n - \frac{3n}{20} \leq n - |x| - |V^2| = |V^{l-2}| .$$

Also

$$|V^l| = n - |x| \geq \tfrac{1}{2}n .$$

Thus by (39),

$$(40) \qquad |x| \leq |V^{l-2}|;\quad |V^2xV^2| \leq \tfrac{1}{4}n;\quad |V^l| \geq \tfrac{1}{2}n .$$

Also, by (39),

$$(41) \qquad \omega = BA,\quad AB = xV^l \quad \text{for some} \quad A, B \in \mathcal{F}^1 .$$

Case 1. $|A| \leqslant |x|$. Set $A = x_2$ so that $x = x_2 x_1$ for some $x_1 \in \mathcal{F}^1$ and $\omega = x_1 V^l x_2$. Since $V \nmid_{\mathrm{i}} x$ and $V \nmid_{\mathrm{f}} x$ we get $V \nmid_{\mathrm{i}} x_2$, $V \nmid_{\mathrm{f}} x_1$. Next if $V^2 \mid_{\mathrm{i}} xV^l$ or $V^2 \mid_{\mathrm{f}} V^l x$ we get $|x| < |V|$ for otherwise $V \mid_{\mathrm{i}} x$ or $V \mid_{\mathrm{f}} x$, a contradiction. Thus in such a case $xV \mid_{\mathrm{i}} V^2$ or $Vx \mid_{\mathrm{f}} V^2$. By Lemma 1.2(iv) we would then have $x \in \langle\langle V \rangle\rangle^1$, a contradiction since $x \neq 1$ and $|x| < |V|$. We have thus shown that $V^2 \nmid_{\mathrm{i}} xV^l$ and $V^2 \nmid_{\mathrm{f}} V^l x$. So $V^5 \nmid_{\mathrm{i}} xV^l$, $V^5 \nmid_{\mathrm{f}} V^l x$. Combined with (39), (40), we see that $\omega \in (V,0,2)$.
Case 2. $|A| = |xV^l|$. Then $\omega = xV^l$. So $\omega = B'A'$, $xV^l = A'B'$, with $A' = 1$. Hence $|A'| \leqslant |x|$ and we are in case 1.
Case 3. The only case left is $|x| < |A| < |xV^l|$. Then by (41), $A \mid_{\mathrm{i}} xV^l$ and by Lemma 1.2(ii)

$$A = xV^j b, \quad ba = V \quad \text{for some} \quad j \in \mathbf{N}, \quad a,b \in \mathcal{F}^1.$$

Hence $|V^j| \leqslant |A| - |x| < |V^l|$ implying $j < l$. So $i = l - j - 1 \in \mathbf{N}$. So $xV^l = xV^j(ba)V^i$ implying by (41) that $B = aV^i$. Hence by (41)

$$(42) \qquad \omega = aV^i x V^j b\,; \quad ba = V\,; \quad i + j + 1 = l\,.$$

Thus if $i, j \geqslant 3$ then by (39), and (40) $\omega \in (V,0,1)$ and we are done. So we are left with the following case

$$(43) \qquad i \leqslant 2 \quad \text{or} \quad j \leqslant 2\,.$$

Now we claim that $V^{i+2} \nmid_{\mathrm{i}} baV^i x$, for otherwise $V^{i+2} \mid_{\mathrm{i}} V^{i+1}x$ implying $V \mid_{\mathrm{i}} x$ a contradiction. So $V^{i+2} \nmid_{\mathrm{i}} baV^i x$. We also claim that $V^{i+2} \nmid_{\mathrm{f}} baV^i x$. For otherwise $V^{i+2} \mid_{\mathrm{f}} V^{i+1}x$. Since $V \nmid_{\mathrm{f}} x$ we get $|x| < |V|$. So $|Vx| < |V^2|$ and $Vx \mid_{\mathrm{f}} V^2$. By Lemma 1.2 (iv), $x \in \langle\langle V \rangle\rangle^1$ a contradiction since $x \neq 1$ and $|x| < |V|$. Hence $V^{i+2} \nmid_{\mathrm{f}} baV^i x$. Similarly $V^{j+2} \nmid_{\mathrm{i}} xV^j ba$, $V^{j+2} \nmid_{\mathrm{f}} xV^j ba$. So

$$(44) \qquad V^{i+2} \nmid_{\mathrm{i}} baV^i x\,; \quad V^{i+2} \nmid_{\mathrm{f}} baV^i x\,; \quad V^{j+2} \nmid_{\mathrm{i}} xV^j ba\,; \quad V^{j+2} \nmid_{\mathrm{f}} xV^j ba\,.$$

Now if $i \leqslant 2$ then $j = l - i - 1 \geqslant 17$ and $|aV^i x| \leqslant |aV^2 x| \leqslant \frac{1}{5}n$. Similarly if $j \leqslant 2$ then $i = l - j - 1 \geqslant 17$ and $|xV^j b| \leqslant \frac{1}{5}n$. So by (42), (43), (44) we see that

$$(45) \qquad \omega = y_1 V^m y_2 \quad \text{for some} \quad y_1, y_2 \in \mathcal{F}^1, \quad m \in \mathbf{Z}^+, \quad m \geqslant 17,$$
$$|y_1| \leqslant \tfrac{1}{5}n, \quad |y_2| \leqslant \tfrac{1}{5}n, \quad V^4 \nmid_{\mathrm{i}} y_2 y_1, \quad V^4 \nmid_{\mathrm{f}} y_2 y_1\,.$$

Now $y_1 = z_1 V^{\nu_1}$, $y_2 = V^{\nu_2} z_2$, for some $\nu_1, \nu_2 \in \mathbf{N}$, $z_1, z_2 \in \mathcal{F}^1$, $V \nmid_{\mathrm{i}} z_2$, $V \nmid_{\mathrm{f}} z_1$. If $V^4 \mid_{\mathrm{i}} z_2 z_1$, then $V^4 \mid_{\mathrm{i}} V^{\nu_2 + 4} \mid_{\mathrm{i}} V^{\nu_2} z_2 z_1 = y_2 z_1 \mid_{\mathrm{i}} y_2 y_1$, a contradiction. Similarly $V^4 \nmid_{\mathrm{f}} z_2 z_1$. So letting $t = m + \nu_1 + \nu_2$ we get,

$$(46) \qquad \begin{cases} \omega = z_1 V^t z_2 \quad \text{for some} \quad z_1, z_2 \in \mathcal{F}^1, \quad t \in \mathbf{Z}^+, \quad t \geqslant 17, \\ |z_1| \leqslant \tfrac{1}{5}n, \quad |z_2| \leqslant \tfrac{1}{5}n; \quad V^4 \nmid_{\mathrm{i}} z_2 z_1, \quad V^4 \nmid_{\mathrm{f}} z_2 z_1, \quad V \nmid_{\mathrm{i}} z_2, \quad V \nmid_{\mathrm{f}} z_1. \end{cases}$$

So

$$(47)\qquad |z_2z_1| \leqslant \frac{2n}{5} \leqslant n - \frac{2n}{5} - \frac{2n}{20} \leqslant n - |z_2z_1| - |V^2| = |V^{t-2}|\,.$$

We next claim that $V^5 \nmid_{\mathrm{i}} z_2z_1V^t$. For suppose $V^5 \mid_{\mathrm{i}} z_2z_1V^t$. Then since $V^4 \nmid_{\mathrm{i}} z_2z_1$ we get $|z_2z_1| < |V^4|$. So $|z_2z_1V| < |V^5|$ and $z_2z_1V \mid_{\mathrm{i}} V^5$ which by Lemma 1.2(iv) implies $z_2z_1 \in \langle\langle V\rangle\rangle^1$. Since $\omega \equiv V^tz_2z_1$ we get by Lemma 1.4 (ii) a contradiction to the fact that ω is primitive. So $V^5 \nmid_{\mathrm{i}} z_2z_1V^t$. Similarly $V^5 \nmid_{\mathrm{f}} V^tz_2z_1$. So by (46) and (47), $\omega \in (V,0,2)$. This proves the lemma.

Definition. Let Γ be a graph and $t \in \mathbf{Z}^+$. Then $\Gamma^{(t)} = (\Gamma^t, —)$ where for $(x_1, ..., x_t), (y_1, ..., y_t) \in \Gamma^{(t)}$ we define $(x_1, ..., x_t) — (y_1, ..., y_t)$ if and only if $(x_1, ..., x_t) = (y_1, ..., y_t)$ or $x_i — y_i$ in Γ where i is the largest integer, $1 \leqslant i \leqslant t$, such that $x_i \neq y_i$.

Remark 4.15. In the above definition notice that $\Gamma^{(1)} = \Gamma$. Also, if $t > 1$ and $z \in \Gamma$ then the induced subgraph $\Gamma_z^{(t)} = \{(x_1, ..., x_{t-1}, z) \mid x_1, ..., x_{t-1} \in \Gamma\}$ of $\Gamma^{(t)}$ is isomorphic to $\Gamma^{(t-1)}$. Moreover if for each $z \in \Gamma$ we choose $x_z \in \Gamma_z^{(t)}$ then $x_z — x_{z'}$ if and only if $z — z'$ in Γ. Hence the induced subgraph $\{x_z \mid z \in \Gamma\}$ of $\Gamma^{(t)}$ is isomorphic to Γ.

Lemma 4.16. *Let Γ be a graph and $r, t \in \mathbf{Z}^+$, $r \leqslant t$. Suppose $\Gamma^{(t)} = P_1 \cup ... \cup P_r$. Then for some $i \in \mathbf{Z}^+$, $1 \leqslant i \leqslant r$, there exists an induced subgraph of $\Gamma^{(t)}$ which is isomorphic to Γ and is also contained in P_i.*

Proof. We can assume, without loss of generality, that $r = t$. Otherwise we can define $P_i = P_1$ for $r < i \leqslant t$. If $t = 1$ there is nothing to prove. We now proceed by induction on t. So let $t > 1$. We use the notation of Remark 4.15. If for some $z \in \Gamma$, $\Gamma_z^{(t)} \cap P_t = \emptyset$ then $\Gamma_z^{(t)} \subseteq P_1 \cup ... \cup P_{t-1}$ and the induced subgraph $\Gamma_z^{(t)}$ is isomorphic to $\Gamma^{(t-1)}$. We are then done by the induction hypothesis. So we may assume that $\Gamma_z^{(t)} \cap P_t \neq \emptyset$ for each $z \in \Gamma$. So for each $z \in \Gamma$ there exists $x_z \in \Gamma_z^{(t)} \cap P_t$. So $\{x_z \mid z \in \Gamma\} \subseteq P_t$. Also the induced subgraph $\{x_z \mid z \in \Gamma\}$ is isomorphic to Γ. This proves the lemma.

Definition. Let Γ be a graph and φ a representation of Γ.

(1) φ is of type 1 if $\varphi(\Gamma) \in \mathcal{D}$ and $\varphi(\Gamma) \subseteq (V,0,1)$ for some primitive $V \in \mathcal{F}$. Then φ restricted to any induced subgraph of Γ is also of type 1.

(2) φ is of type 2 if $\varphi(\Gamma) \in \mathcal{D}$ and $\varphi(\Gamma) \subseteq (V,0,2)$ for some primitive $V \in \mathcal{F}$. Then φ restricted to any induced subgraph of Γ is also of type 2.

Lemma 4.17. *Assume (*). Then either every finite graph has a representation of type 1 or every finite graph has a representation of type 2.*

Proof. Suppose there exists a finite graph Φ such that Φ has no representation of type 1. Let Γ_1 be any finite graph and set $\Gamma = \Gamma_1 + \Phi$. So Γ has no representation of type 1. By Lemma 4.13 there exists a representation φ of $\Gamma^{(2)}$, $V \in \mathcal{F}$ primitive such that $\varphi(\Gamma^{(2)}) \in \mathcal{D}$ and $\varphi(\Gamma^{(2)}) \subseteq (V, 20)$. By Lemma 4.14, $\varphi(\Gamma^{(2)}) \subseteq (V,0,1) \cup (V,0,2)$. Then Lemma 4.16 implies that there exists an induced subgraph Γ' of $\Gamma^{(2)}$ such that Γ' is isomorphic to Γ and $\varphi(\Gamma') \subseteq (V,0,1)$ or $\varphi(\Gamma') \subseteq (V,0,2)$. Since Γ has no representation of type 1 and $\varphi(\Gamma^{(2)}) \in \mathcal{D}$ we get $\varphi(\Gamma') \subseteq (V,0,2)$ and $\varphi(\Gamma') \in \mathcal{D}$. So Γ' has a representation of type 2. Since Γ_1 is an induced subgraph of Γ and Γ is isomorphic to Γ' we conclude that Γ_1 has a representation of type 2. This proves the lemma.

Lemma 4.18. *Let $x, y, V \in \mathcal{F}$, $S, T \in \mathcal{F}^1$, $k \in \mathbf{Z}^+$, $k \geqslant 5$. Suppose further that V is primitive, $V \nmid_i x$, $V \nmid_i y$, $V \nmid_f y$, $V \nmid_f x$ and $|x| \leqslant |y| \leqslant |V^{k-4}|$. Then*

(i) *If $SV^2yV^2T = V^kxV^k$ then either $Vy = xV$, or $yV = Vx$, or else $S = V^{k-2} = T$ and $x = y$.*

(ii) *If $x = y$ and $SV^2xV^2T = V^kxV^k$ then $S = V^{k-2} = T$.*

(iii) *If $x = y$ and $SV^2xV^2T = (V^kx)^3$ then either $S = V^{k-2}$ or $S = V^kxV^{k-2}$.*

(iv) *If $V^2yV^2 \to V^kx$ then either $Vx = yV$, or $xV = Vy$, or else $x = y$.*

Proof. The hypothesis implies that $x \notin \langle\langle V \rangle\rangle^1$ and $y \notin \langle\langle V \rangle\rangle^1$.

(i) First assume $|SV^2| \leqslant |V^k|$ and $yV \neq Vx$. We will show that $S = V^{k-2} = T$ and $x = y$. Now $SV^2 \mid_i V^k$. By Lemma 1.2 (iv), $S \in \langle\langle V \rangle\rangle^1$. Hence $S = V^j$ for some $j \in \mathbf{N}$ and $j + 2 \leqslant k$. So

$$(48) \qquad S = V^j; \quad yV^2T = V^lxV^k; \quad j, l = k - j - 2 \in \mathbf{N}.$$

We claim that $l = 0$. For suppose $l \geqslant 1$. Then since $V \nmid_i y$ we get $|y| < |V|$. If $l \geqslant 2$ then $|yV| < |V^2| \leqslant |V^l|$ and $yV \mid_i V^l$ implying by Lemma 1.2 (iv) that $y \in \langle\langle V \rangle\rangle^1$, a contradiction. So $l = 1$ and $yV^2T = VxV^k$. Since $|x| \leqslant |y|$ we get $|Vx| \leqslant |yV|$ and $yV = Vx\alpha$ for some $\alpha \in \mathcal{F}^1$. Further $|\alpha| = |y| - |x| < |y| < |V|$. We have $\alpha VT = V^k$. By Lemma 1.2 (iv), $\alpha \in \langle\langle V \rangle\rangle^1$. Since $|\alpha| < |V|$ we have $\alpha = 1$ implying $yV = Vx$ contrary to our assumption. So in (48) we must have $l = 0$ and

$$(49) \qquad yV^2T = xV^k, \quad S = V^{k-2}.$$

Since $|y| \geqslant |x|$ we get $y = x\beta$ for some $\beta \in \mathcal{F}^1$. Hence $\beta V^2T = V^k$ and by Lemma 1.2 (iv) $\beta \in \langle\langle V \rangle\rangle^1$. Since $\beta \mid_f y$ and $V \nmid_f y$ it must be that $\beta = 1$. Hence $x = y$ and by (49), $S = V^{k-2} = T$.

By a dual argument we see that $|V^2T| \leqslant |V^k|$ and $Vy \neq xV$ imply $S = T = V^{k-2}$ and $x = y$. So we are left with the case $|V^2T| > |V^k|$, $|SV^2| > |V^k|$. But then

$$(50) \qquad |x| = |SV^2| - |V^k| + |y| + |V^2T| - |V^k| > |y|.$$

This contradiction proves (i).

(ii) Since V is primitive and $x \notin \langle\langle V \rangle\rangle^1$ we get $Vx \neq xV$. We are now done by (i).

(iii) We have

$$(51) \qquad SV^2xV^2T = V^kxV^kxV^kx.$$

If $|SV^2xV^2| \leqslant |V^kxV^k|$ then $SV^2xV^2 \mid_i V^kxV^k$ and by (ii) $S = V^{k-2}$. So let $|SV^2xV^2| > |V^kxV^k|$. Then

$$|S| > |V^k| + |V^{k-4}| \geqslant |V^kx|\,.$$

By (51) we get

$$(52) \qquad S = V^kxS' \quad \text{for some } S' \in \mathcal{F};\; S'V^2xV^2T = V^kxV^kx\,.$$

If $|S'V^2xV^2| \leqslant |V^kxV^k|$ then $S'V^2xV^2 \mid_i V^kxV^k$ and by (ii) $S' = V^{k-2}$. So by (52) $S = V^kxV^{k-2}$ and we are done. So we are left with the case $|S'V^2xV^2| > |V^kxV^k|$. Hence

$$|S'| > |V^k| + |V^{k-4}| \geqslant |V^kx|\,.$$

By (52), we get

$$(53) \qquad S' = V^kxS'' \quad \text{for some } S'' \in \mathcal{F};\; S''V^2xV^2T = V^kx\,.$$

But then $S''V^2xV^2TV^k = V^kxV^k$ and by (ii), $S'' = V^{k-2}$. So by (53) $V^kxV^2T = V^kx$, a contradiction. This proves (iii).

(iv) Since $|y| \leqslant |V^{k-4}|$ we get

$$(54) \qquad |V^2yV^2| \leqslant |V^k|\,.$$

Since $x \neq 1$ we get $V^kx \nmid V^2yV^2$. By Lemma 1.3(v) $V^2yV^2 \mid (V^kx)^2$. By Lemma 1.3 (ii),

$$(55) \qquad SV^2yV^2T = V^kxV^kx \quad \text{for some} \quad S, T \in \mathcal{F}^1, \quad |S| < |V^kx|\,.$$

By (54), (55) we get

$$|SV^2yV^2| = |S| + |V^2yV^2| < |V^kxV^k|\,.$$

So by (55), $|x| < |T|$ and $T = T'x$ for some $T' \in \mathcal{F}$. Hence $SV^2yV^2T' = V^kxV^k$. We are now done by (i).

Lemma 4.19. *Let* $\omega_1, \omega_2, x, V \in \mathcal{F}$, $a_1, b_1, a_2, b_2 \in \mathcal{F}^1$, $i_1, i_2, j_1, j_2 \in \mathbf{Z}^+$. *Set* $A_1 = a_1V^{i_1}$, $A_2 = a_2V^{i_2}$, $B_1 = V^{j_1}b_1$, $B_2 = V^{j_2}b_2$. *Suppose that* V *is primitive,* $V \nmid_i x$, $V \nmid_f x$, $\omega_1 = A_1xB_1$, $\omega_2 = A_2xB_2$, $\omega_2 \nmid \omega_1$, $b_1a_1 = b_2a_2 = V$. *Assume further that* $i_1, j_1, i_2, j_2 \geqslant 2$ *and* $|V^2xV^2| \leqslant |V^{i_2+j_2+1}|$. *Then* $\omega_1 \to \omega_2$ *if and only if* $B_1 \mid_i B_2\omega_2$ *and* $A_1 \mid_f \omega_2A_2$.

Proof. Suppose first that $B_1 \mid_i B_2\omega_2$, $A_1 \mid_f \omega_2A_2$. Then by Lemma 1.2 (i), $\omega_1 = A_1xB_1 \mid \omega_2A_2xB_2\omega_2 = \omega_2^3$ whence $\omega_1 \to \omega_2$. Conversely let $\omega_1 \to \omega_2$. By Lemma 1.3 (v), $\omega_1 \mid \omega_2^2$. Let $k = i_2 + j_2 + 1 \geqslant 5$. Then $\omega_2 \equiv V^kx$ whence $\omega_2^2 \mid (V^kx)^3$. So $\omega_1 \mid (V^kx)^3$ and

$$S_1a_1V^{i_1}xV^{j_1}b_1T_1 = (V^kx)^3 \quad \text{for some} \quad S_1, T_1 \in \mathcal{F}^1\,.$$

Let $S = S_1 a_1 V^{i_1-2}$, $T = V^{j_1-2} b_1 T_1$. Then

(56) $$SV^2xV^2T = (V^kx)^3 .$$

Also,

$$A_1 = a_1 V^{i_1} \mid_{\mathrm{f}} S_1 a_1 V^{i_1-2} V^2 = SV^2 ;$$

$$B_1 = V^{j_1} b_1 \mid_{\mathrm{i}} V^{j_1} b_1 T_1 = V^2 T .$$

Thus

(57) $$A_1 \mid_{\mathrm{f}} SV^2; \quad B_1 \mid_{\mathrm{i}} V^2T .$$

Since $|V^2xV^2| \leqslant |V^k|$ we get $|x| \leqslant |V^{k-4}|$. Thus by Lemma 4.18 (iii) $S = V^{k-2}$ or $S = V^kxV^{k-2}$. By (56), $T = V^{k-2}xV^kx$ or $T = V^{k-2}x$. So in any case

$$SV^2 \mid_{\mathrm{f}} V^kxV^k \mid_{\mathrm{f}} \omega_2^2 A_2 ; \quad V^2T \mid_{\mathrm{i}} V^kxV^kx \mid_{\mathrm{i}} B_2\omega_2^2 .$$

So by (57)

$$A_1 \mid_{\mathrm{f}} \omega_2^2 A_2 ; \quad B_1 \mid_{\mathrm{i}} B_2 \omega_2^2 .$$

If $|A_1| \geqslant |\omega_2 A_2|$ or $|B_1| \geqslant |B_2\omega_2|$, then $\omega_2 \mid A_1$ or $\omega_2 \mid B_1$ implying $\omega_2 \mid \omega_1$, a contradiction. So $|A_1| < |\omega_2 A_2|$ and $|B_1| < |B_2\omega_2|$. Hence $A_1 \mid_{\mathrm{f}} \omega_2 A_2$ and $B_1 \mid_{\mathrm{i}} B_2\omega_2$ proving the lemma.

Lemma 4.20. *There exists a finite graph which has no representation of type* 1.

Proof. We assume that every finite graph has a representation of type 1 and obtain a contradiction. Let $I = \{0,1\}$, $J = \{1,2,3\}$ and let $\Gamma = I \times J$ where for $(i,j), (i',j') \in \Gamma$ we define $(i,j) — (i',j')$ if and only if $(i,j) = (i',j')$ or $j \neq j'$. Let $\xi \notin \Gamma$ and let φ be a representation of $\Gamma + \xi$ of type 1. Then for some primitive $V \in \mathcal{F}$, $\varphi(\Gamma + \xi) \subseteq (V,0,1)$ and $\varphi(\Gamma + \xi) \in \mathcal{D}$. Let $\omega = \varphi(\xi)$ and $\mathcal{A} = \varphi(\Gamma)$. Then $u — \omega$ for all $u \in \mathcal{A}$. Since $\varphi(\Gamma + \xi) \in \mathcal{D}$ we get $u \nmid \omega$, $\omega \nmid u$, $|u| < 2|\omega|$, $|\omega| < 2|u|$, for all $u \in \mathcal{A}$. Let $u \in \mathcal{A}$. Then since $u, \omega \in (V,0,1)$, we have

(58)
$$\omega = cV^lxV^md; \quad u = aV^{l_1}yV^{m_1}b; \quad V \nmid_{\mathrm{i}} x, \quad V \nmid_{\mathrm{f}} x, \quad V \nmid_{\mathrm{i}} y, \quad V \nmid_{\mathrm{f}} y;$$
$$ba = dc = V; \quad a,b,c,d \in \mathcal{F}^1, \quad x,y \in \mathcal{F},$$
$$l,m,l_1,m_1 \in \mathbf{Z}^+, \quad l,m,l_1,m_1 \geqslant 3, \quad |V^2xV^2| \leqslant \tfrac{1}{4}|\omega|,$$
$$|V^2yV^2| \leqslant \tfrac{1}{4}|u|, \quad |\omega| \leqslant 2|V^{l+m+1}|, \quad |u| \leqslant 2|V^{l_1+m_1+1}| .$$

First assume $|y| \leqslant |x|$. Then since $V^2xV^2 \mid \omega — u$ we get $V^2xV^2 \to u$. Let $k_1 = l_1 + m_1 + 1 \geqslant 5$. Then $u \equiv V^{k_1}y$ whereby $V^2xV^2 \to V^{k_1}y$. Moreover,

$$|V^2xV^2| \leqslant \tfrac{1}{4}|\omega| < \tfrac{1}{2}|u| \leqslant |V^{k_1}| .$$

Hence $|y| \leqslant |x| \leqslant |V^{k_1-4}|$ and by (58) and Lemma 4.18(iv) we get $|x| = |y|$.

Thus in any case we can assume $|x| \leqslant |y|$. Then $V^2yV^2 | u \to \omega$ implying

$V^2yV^2 \to \omega$. Let $k = l + m + 1 \geqslant 5$. So $\omega \equiv V^k x$ whereby $V^2 y V^2 \to V^k x$. Moreover

$$|V^2yV^2| \leqslant \tfrac{1}{4}|u| < \tfrac{1}{2}|\omega| \leqslant |V^k|.$$

Hence $|x| \leqslant |y| \leqslant |V^{k-4}|$. So by (58) and Lemma 4.18 (iv) we have that either $x = y$ or $Vx = yV$ or $xV = Vy$. We see by (58) that in any case we can write $u = aV^i x V^j b$, $i,j \in \mathbf{Z}^+$, $i,j \geqslant 2$, $i+j+1 = l_1 + m_1 + 1$. Moreover,

$$|V^2xV^2| = |V^2yV^2| \leqslant \tfrac{1}{4}|u| < |V^{l_1+m_1+1}| = |V^{i+j+1}|.$$

To summarize, there exists a fixed $x \in \mathcal{F}$ such that $V \nmid_{\mathrm{i}} x$, $V \nmid_{\mathrm{f}} x$ and so that every $u \in \mathcal{A}$ admits a decomposition of the following type

$$u = AxB;\ A = aV^i,\ B = V^j b,\ ba = V;\ a,b \in \mathcal{F}^1,\ i,j \in \mathbf{Z}^+; \tag{59}$$
$$i,j \geqslant 2,\ |V^2xV^2| \leqslant |V^{i+j+1}|.$$

We fix one such decomposition for each $u \in \mathcal{A}$. Let $u_1, u_2 \in \mathcal{A}$ and corresponding to (59) let $u_1 = A_1xB_1$, $u_2 = A_2xB_2$. If $u_1 \neq u_2$ then since $\mathcal{A} \in \mathcal{D}$ we have that $u_2 \nmid u_1$. So by Lemma 4.19 $u_1 \to u_2$ if and only if $B_1 |_{\mathrm{i}} B_2u_2$ and $A_1 |_{\mathrm{f}} u_2A_2$. If $u_1 = u_2$ then the same holds trivially since we are fixing one decomposition for each word in $\mathcal{A}$. So for any $u_1, u_2 \in \mathcal{A}$,

$$u_1 \to u_2 \quad \text{if and only if} \quad B_1 |_{\mathrm{i}} B_2u_2 \quad \text{and} \quad A_1 |_{\mathrm{f}} u_2A_2, \tag{60}$$

where in (59), $u_1 = A_1xB_1$ and $u_2 = A_2xB_2$.

Next, with respect to (59) choose $u_1 = A_1xB_1 \in \mathcal{A}$ with $|A_1|$ maximal. Similarly choose $u_2 = A_2xB_2 \in \mathcal{A}$ with $|B_2|$ maximal. We have

$$u_1 = \varphi(\lambda_1, t_1),\ u_2 = \varphi(\lambda_2, t_2) \quad \text{for some} \quad \lambda_1, \lambda_2 \in I,\ t_1, t_2 \in J. \tag{61}$$

We can choose $t \in J$ such that $t \neq t_1$, $t \neq t_2$. Let

$$\begin{aligned} \varphi(0,t) &= u_3 = A_3xB_3 \quad \text{with respect to (59)},\\ \varphi(1,t) &= u_4 = A_4xB_4 \quad \text{with respect to (59)}. \end{aligned} \tag{62}$$

By maximality of $|A_1|$ and $|B_2|$ we have

$$|A_3|, |A_4| \leqslant |A_1|;\quad |B_3|, |B_4| \leqslant |B_2|. \tag{63}$$

Since $t \neq t_1$, $t \neq t_2$ we have by (61), (62),

$$u_1 \text{---} u_3,\quad u_2 \text{---} u_3,\quad u_1 \text{---} u_4,\quad u_2 \text{---} u_4. \tag{64}$$

By (60), (63), (64) we get

$$A_3 |_{\mathrm{f}} A_1 |_{\mathrm{f}} u_4A_4;\quad A_4 |_{\mathrm{f}} A_1 |_{\mathrm{f}} u_3A_3;$$
$$B_3 |_{\mathrm{i}} B_2 |_{\mathrm{i}} B_4u_4;\quad B_4 |_{\mathrm{i}} B_2 |_{\mathrm{i}} B_3u_3.$$

So we get

$$A_3 |_{\mathrm{f}} u_4A_4,\quad B_3 |_{\mathrm{i}} B_4u_4,\quad A_4 |_{\mathrm{f}} u_3A_3,\quad B_4 |_{\mathrm{i}} B_3u_3.$$

So by (60), u_3—u_4. Hence by (62), $(0,t)$—$(1,t)$, a contradiction. This proves the lemma.

Lemma 4.21. *Let $V, \omega \in \mathcal{F}$, $x_1, x_2 \in \mathcal{F}^1$, $i \in \mathbf{Z}^+$. Suppose that V is primitive, $i \geqslant 3$, $\omega = x_1 V^i x_2$, $V \not|_{\mathrm{i}}\, x_2$, $V \not|_{\mathrm{f}}\, x_1$, $V^5 \not|_{\mathrm{i}}\, x_2 x_1 V^i$, $V^5 \not|_{\mathrm{f}}\, V^i x_2 x_1$, $|x_2 x_1| \leqslant |V^{i-2}|$. Let $\mathcal{X} = \{(S,T) \mid S, T \in \mathcal{F}^1,\ SV^iT = \omega^2\}$. Then*

(i) $|\mathcal{X}| \leqslant 10$.

(ii) $V^{i+10} \not| \,\omega^2$.

Proof. (i) For $(S,T) \in \mathcal{X}$ set $\delta(S,T) = S$, $\rho(S,T) = T$. If $(S,T), (S',T') \in \mathcal{X}$ and $\delta(S,T) = \delta(S',T')$ then $S = S'$ and $SV^iT = \omega^2 = S'V^iT'$. Hence $T = T'$ and δ is one to one. Similarly ρ is injective.

Next let $(S,T) \in \mathcal{X}$. Then

(65) $$SV^iT = x_1 V^i x_2 x_1 V^i x_2 .$$

First assume $|S| < |x_1|$. Then $x_1 = Sy$ for some $y \in \mathcal{F}$ and

$$V^iT = yV^i x_2 x_1 V^i x_2 .$$

Also

$$|yV| \leqslant |x_2 x_1 V| < |V^i| .$$

Hence $yV \mid_{\mathrm{i}} V^i$. By Lemma 1.2 (iv), $y \in \langle\langle V \rangle\rangle^1$. Since $y \neq 1$ we get $y \in \langle\langle V \rangle\rangle$. Hence $V \mid_{\mathrm{f}} y \mid_{\mathrm{f}} x_1$ implying $V \mid_{\mathrm{f}} x_1$, a contradiction. Thus $|x_1| \leqslant |S|$. Similarly $|x_2| \leqslant |T|$. Hence

(66) $$S = x_1 S', \quad T = T' x_2 \quad \text{for some} \quad S', T' \in \mathcal{F}^1 .$$

By (65) we get,

(67) $$S'V^iT' = V^i x_2 x_1 V^i .$$

First assume $|S'V| \leqslant |V^i|$. Then $S'V \mid_{\mathrm{i}} V^i$ and by Lemma 1.2 (iv), $S' = V^j$ for some $j \in \mathbf{N}$. Hence

$$V^{i+j}T' = V^i x_2 x_1 V^i .$$

Thus $V^jT' = x_2 x_1 V^i$ and $V^j \mid_{\mathrm{i}} x_2 x_1 V^i$. By hypothesis $j \leqslant 4$. Also $S = x_1 V^j$. Hence

(68) $$|S'V| \leqslant |V^i| \quad \text{implies} \quad S \in \{x_1 V^l \mid l = 0, 1, \ldots, 4\} = X .$$

By a dual argument

(69) $$|VT'| \leqslant |V^i| \quad \text{implies} \quad T \in \{V^l x_2 \mid l = 0, \ldots, 4\} = Y .$$

The only case left is when $|S'V| > |V^i|$ and $|VT'| > |V^i|$. Then by (67),

$$|x_2 x_1| = |S'| + |T'| - |V^i| > |V^{i-1}| + |V^{i-1}| - |V^i| = |V^{i-2}| .$$

Hence $|x_2 x_1| > |V^{i-2}|$, a contradiction. Hence by (68), (69), $u \in \mathcal{X}$ implies $\delta(u) \in X$ or $\rho(u) \in Y$. Since $|X| = |Y| = 5$ and ρ and δ are injections we see that $|\mathcal{X}| \leqslant 10$.

(ii) Suppose $V^{i+10} \mid \omega^2$. Then $SV^{i+10}T = \omega^2$ for some $S, T \in \mathcal{F}^1$. Then for $j \in \mathbf{N}$, $0 \leqslant j \leqslant 10$, $(SV^j)V^i(V^{10-j})T = \omega^2$.
Hence $(SV^j, V^{10-j}T) \in \mathcal{X}$ for $j = 0, 1, \ldots, 10$. This contradicts (i).

Lemma 4.22. *There exists a finite graph which has no representation of type* 2.

Proof. We assume that every finite graph has a representation of type 2 and obtain a contradiction. Let $\Lambda = \Lambda_{10}$ be the graph in Construction 4.7. Let $\Gamma = \Lambda^{(100)} + \Lambda^{(100)}$ and let θ be a representation of Γ of type 2. Then $\mathcal{A} = \theta(\Gamma) \in \mathcal{D}$ and there exists $V \in \mathcal{F}$ primitive such that $\mathcal{A} \subseteq (V, 0, 2)$. So for each $\omega \in \mathcal{A}$,

$$\begin{aligned}(70)\qquad &\omega = x_\omega V^\alpha y_\omega,\quad \alpha = \alpha(\omega) \in \mathbf{Z}^+,\quad \alpha \geqslant 3,\quad x_\omega, y_\omega \in \mathcal{F}^1,\quad V \nmid_{\mathrm{i}} y_\omega,\\ &V \nmid_{\mathrm{f}} x_\omega,\quad V^5 \nmid_{\mathrm{i}} y_\omega x_\omega V^\alpha,\quad V^5 \nmid_{\mathrm{f}} V^\alpha y_\omega x_\omega,\quad |y_\omega x_\omega| \leqslant |V^{\alpha-2}|.\end{aligned}$$

Choose $\omega_0 \in \mathcal{A}$ such that $j = \alpha(\omega_0)$ is minimal. Let $\omega_0 = \theta(\mu, \lambda)$ where $\mu \in \Lambda^{(100)}$ and $\lambda \in \{0,1\}$. Consider the representation φ of $\Lambda^{(100)}$ given by $\varphi(\nu) = \theta(\nu, 1-\lambda)$. Let $\mathcal{B} = \varphi(\Lambda^{(100)})$. Then $\mathcal{B} \subseteq \mathcal{A}$ whence $\mathcal{B} \in \mathcal{D}$. For all $u \in \mathcal{B}$, $u — \omega_0$. Then if in (70) $\alpha(u) = k$, then $V^k \mid u$. Since $\mathcal{A} \in \mathcal{D}$, $u \mid \omega_0^2$. Hence $V^k \mid \omega_0^2$. Now by Lemma 4.21 (ii), $V^{j+10} \nmid \omega_0^2$. Hence $k \leqslant j + 9$. By minimality of j we have $j \leqslant k$ whereby

$$(71)\qquad \alpha(u) \in L = \{j+0, j+1, \ldots, j+9\} \quad \text{for all} \quad u \in \mathcal{B}.$$

Now for $u \in \mathcal{B}$, $u = x_u V^{\alpha(u)} y_u$ and since $u \mid \omega_0^2$ we get

$$(72)\qquad \omega_0^2 = a_u x_u V^{\alpha(u)} y_u b_u \quad \text{for some} \quad a_u, b_u \in \mathcal{F}^1.$$

Let $\mathcal{X} = \{(S,T) \mid S, T \in \mathcal{F}^1,\ SV^jT = \omega_0^2\}$. By Lemma 4.21, $|\mathcal{X}| \leqslant 10$. For $u \in \mathcal{B}$, we see by (71), (72) that $\delta(u) = (a_u x_u V^{\alpha(u)-j}, y_u b_u) \in \mathcal{X}$. So $\delta_1(u) = (\alpha(u), \delta(u)) \in L \times \mathcal{X}$ for all $u \in \mathcal{B}$. Hence $|L \times \mathcal{X}| \leqslant 100$. Also

$$\mathcal{B} = \bigcup_{\xi \in L \times \mathcal{X}} \delta_1^{-1}(\xi).$$

Since $\mathcal{B} = \varphi(\Lambda^{(100)})$, we see by Lemma 4.16 that there exists an induced subgraph Λ' of $\Lambda^{(100)}$ such that Λ' is isomorphic to Λ and $\mathcal{C} = \varphi(\Lambda') \subseteq \delta_1^{-1}(\xi)$ for some $\xi \in L \times \mathcal{X}$. Since $\mathcal{C} \subseteq \mathcal{B}$ we get $\mathcal{C} \in \mathcal{D}$. Hence there exists a representation ψ of Λ such that $\psi(\Lambda) = \mathcal{C} \subseteq \delta_1^{-1}(\xi)$. So for all $u, v \in \mathcal{C}$, $\alpha(u) = \alpha(v)$ and $(a_u x_u V^{\alpha(u)-j}, y_u b_u) = (a_v x_v V^{\alpha(v)-j}, y_v b_v)$. Hence there exists $t \in \mathbf{Z}^+$ such that $t = \alpha(u)$ for all $u \in \mathcal{C}$ and for all $u, v \in \mathcal{C}$, $a_u x_u = a_v x_v$ and $y_u b_u = y_v b_v$. Hence

$$\begin{aligned}(73)\qquad &u = x_u V^t y_u \quad \text{for all} \quad u \in \mathcal{C};\quad a_u x_u = a_v x_v,\\ &y_u b_u = y_v b_v \quad \text{for all} \quad u, v \in \mathcal{C}.\end{aligned}$$

For $u, v \in \mathcal{C}$ we have by (73) that $x_u \mid_{\mathrm{f}} x_v$ or $x_v \mid_{\mathrm{f}} x_u$. We define $u \leqslant v$ if and only if $x_u \mid_{\mathrm{f}} x_v$. Evidently $\leqslant$ is transitive and for all $u, v \in \mathcal{C}$, $u \leqslant v$ or $v \leqslant u$. Let $u \leqslant v \leqslant u$. Then $x_u = x_v$. By (73) $u = x_u V^t y_u = x_v V^t y_u$, $v = x_v V^t y_v$. Now by (73)

$y_u \mid_i y_v$ or $y_v \mid_i y_u$ implying $u \mid_i v$ or $v \mid_i u$. Since $\mathcal{E} \in \mathcal{D}$ we get $u = v$. Thus $\leqslant$ is a linear order on $\mathcal{E}$. Now again let $u, v \in \mathcal{E}$, $u < v$. Then it cannot be that $y_u \mid_i y_v$. For otherwise we get by (73) and Lemma 1.2(i) that $u = x_u V^t y_u \mid x_v V^t y_v = v$. Since $\mathcal{E} \in \mathcal{D}$ we get $u = v$, a contradiction. Thus by (73) $y_v \mid_i y_u$. Hence

(74) $\quad u, v \in \mathcal{E}, \quad u < v \quad$ implies $\quad x_u \mid_f x_v \quad$ and $\quad y_v \mid_i y_u$.

Let $\omega_1, \omega_2, \omega_3 \in \mathcal{E}$, $\omega_1 < \omega_2 < \omega_3$. Then by (73), (74),

$$x_{\omega_1} \mid_f x_{\omega_2} \mid_f x_{\omega_3}, \quad y_{\omega_3} \mid_i y_{\omega_2} \mid_i y_{\omega_1}.$$

Thus by (73) and Lemma 1.2(i) we get

(75) $\quad \omega_2 \mid x_{\omega_3} V^t y_{\omega_1} \quad$ for any $\quad \omega_1, \omega_2, \omega_3 \in \mathcal{E} \quad$ such that $\quad \omega_1 < \omega_2 < \omega_3$.

Now since $\mathcal{E} = \psi(\Lambda)$ and $\Lambda = \Lambda_{10}$ we see by Lemma 4.10 that there exist distinct elements $u_1, \ldots, u_{21}$, $v \in \mathcal{E}$ such that

(76) $\quad u_1 < u_2 < \ldots < u_{21}; \quad u_i — v$ if i is odd,

$\quad\quad u_i \nrightarrow v$ for i even $(1 \leqslant i \leqslant 21,\ i \in \mathbf{Z}^+)$.

Let $M = \{1, 3, 5, \ldots, 21\}$. Then $|M| = 11$. Since $\mathcal{E} \in \mathcal{D}$, we get by (76) that $u_i \mid v^2$ for $i \in M$. Hence for each $i \in M$ there exist $A_i, B_i \in \mathcal{F}^1$ such that $A_i u_i B_i = v^2$. Thus by (73),

(77) $\quad A_i x_{u_i} V^t y_{u_i} B_i = v^2 \quad$ for all $\quad i \in M$.

Let $\mathcal{L} = \{(S, T) \mid S, T \in \mathcal{F}^1,\ S V^t T = v^2\}$. By (70), (73) and Lemma 4.21, $|\mathcal{L}| \leqslant 10$. Now by (77), $(A_i x_{u_i}, y_{u_i} B_i) \in \mathcal{L}$, for all $i \in M$. Since $|M| = 11$ there exist $i, l \in M$, $i < l$ such that $(A_i x_{u_i}, y_{u_i} B_i) = (A_l x_{u_l}, y_{u_l} B_l)$. Hence by (77)

(78) $\quad v^2 = A_l x_{u_l} V^t y_{u_i} B_i$.

Since i and l are odd, $k = i + 1 < l$ and k is even. Hence $u_i < u_k < u_l$. By (75)

$$u_k \mid x_{u_l} V^t y_{u_i} .$$

So by (78) $u_k \mid v^2$ whereby $u_k \to v$. But this contradicts (76) proving the lemma.

Combining Lemmas 4.17, 4.20 and 4.22 we see that (*) is false. We have thus proved the following theorem.

Theorem 4.23. *There exists a finite graph not contained in the free semi-group.*

Remark 4.24. While our proof of Theorem 4.23 was not formally constructive, a closer examination of the proof shows that we actually have produced a concrete finite graph not contained in the free semigroup. However, the order of this graph is astronomically large (between 10^{10^α} and 10^{10^β} where $\alpha = 10^9$, $\beta = 10^{10}$). The problem of finding 'small' graphs not contained in the free semigroup remains open.

5. Equations of the archimedean relation

In this section we study the archimedean relation on a free semigroup by establishing a correspondence with the solutions of a particular equation. While special equations in the free semigroup have been studied for a long time by various authors, a systematic study of equations in the free semigroup was begun only recently by André Lentin [3, 15, 16]. We start by taking care of some preliminary cases.

Theorem 5.1. *Let $\omega_1, \omega_2 \in \mathcal{F}$. Then $\omega_1 \mid \omega_2$ and $\omega_1 — \omega_2$, if and only if one of the following is true.*

(i) *There exist $a, b, c \in \mathcal{F}^1$, $i \in \mathbf{Z}^+$ such that $\omega_1 = abc$ and $\omega_2 = bc(abc)^i ab$,*

or

(ii) *There exist $a, b, c \in \mathcal{F}^1$, $i \in \mathbf{Z}^+$ such that $\omega_1 = abc$ and $\omega_2 = c(abc)^i a$.*

Proof. The converse being trivial, let $\omega_1 \mid \omega_2$ and $\omega_1 — \omega_2$. So $|\omega_1| \leqslant |\omega_2|$. If $|\omega_1| = |\omega_2|$ then $\omega_1 = \omega_2$ and (ii) holds trivially with $a = c = i = 1$. So let $|\omega_1| < |\omega_2|$. Now $\omega_1 = U^t$ for some $t \in \mathbf{Z}^+$ and some primitive $U \in \mathcal{F}$. Then $\omega_2 \to U$. Since $|U| < |\omega_2|$ we get $\omega_2 \not\mid U$. By Lemma 1.3(i) there exist $k \in \mathbf{N}$, $x, y \in \mathcal{F}^1$ such that $x \mid_f U$, $y \mid_i U$, $|x| < |U|$, $|y| < |U|$ and $\omega_2 = xU^k y$. There exist $x_1, y_1 \in \mathcal{F}$ such that $x_1 x = U = yy_1$. Since $U^t = \omega_1 \mid \omega_2$ we get $u_1 U^t v_1 = xU^k y$ for some $u_1, v_1 \in \mathcal{F}^1$. So $(x_1 u_1) U^t (v_1 y_1) = x_1 x U^k y y_1 = U^{k+2}$. Since $x_1 \neq 1$ and $y_1 \neq 1$, Lemma 1.2 (iv) implies that $x_1 u_1, v_1 y_1 \in \langle\langle U \rangle\rangle$. So $|U^{k+2}| = |(x_1 u_1) U^t (v_1 y_1)| \geqslant |U^{t+2}|$. This implies that $t \leqslant k$. Hence $k = it + j$ for some $i \in \mathbf{Z}^+$, $j \in \mathbf{N}$, $j \leqslant t - 1$. So

$$\omega_2 = xU^k y = xU^{it} U^j y = x\omega_1^i U^j y .$$

Let $z = U^j y$. Then $z = U^j y \mid_i U^{j+1} \mid_i U^t = \omega_1$. Moreover $x \mid_f U \mid_f \omega_1$. So

$$\omega_2 = x\omega_1^i z, \quad z \mid_i \omega_1, \quad x \mid_f \omega_1, \quad i \in \mathbf{Z}^+. \tag{79}$$

Hence $ux = zv = \omega_1$ for some $u, v \in \mathcal{F}^1$. Let us first assume $|u| \leqslant |z|$. Then $z = ua$ for some $a \in \mathcal{F}^1$. So $x = av$ and $\omega_1 = uav$. By (79) we get

$$\omega_1 = uav, \quad \omega_2 = av(uav)^i ua , \quad i \in \mathbf{Z}^+.$$

Thus (i) holds. Next assume $|z| < |u|$. Then $u = zb$ for some $b \in \mathcal{F}^1$. Then $\omega_1 = zbx$. So by (79),

$$\omega_1 = zbx, \quad \omega_2 = x(zbx)^i z, \quad i \in \mathbf{Z}^+ .$$

So (ii) holds. This proves the theorem.

Definition. (1) The class $\mathfrak{T} = \{(\omega_1,\omega_2) | \omega_1, \omega_2 \in \mathcal{F}(\Omega)$ for some non-empty set Ω such that $|\omega_1| \leq |\omega_2|$ and $\omega_1 — \omega_2\}$.

(2) The class $\mathfrak{T}_0 = \{(\omega_1,\omega_2) | (\omega_1,\omega_2) \in \mathfrak{T}$ and $\omega_1 \mid \omega_2\}$.

Note that Theorem 5.1 completely characterizes $\mathfrak{T}_0$.

Lemma 5.2. *Let $u, v, \omega \in \mathcal{F}^1$ such that $uv\omega | uv^2\omega$. Then either $uv\omega \mid_i uv^2\omega$ or $uv\omega \mid_f uv^2\omega$.*

Proof. If $v = 1$ then $uv\omega = uv^2\omega$ and there is nothing to prove. So let $v \neq 1$. Then $v = V^t$ for some $t \in \mathbf{Z}^+$, $V \in \mathcal{F}$ such that V is primitive. Since $uv\omega | uv^2\omega$ we get

$$Suv\omega T = uv^2\omega \quad \text{for some} \quad S, T \in \mathcal{F}^1.$$

If $S = 1$, then there is nothing to prove . So let $S \neq 1$. Let $u = xV^\alpha$, $x \in \mathcal{F}^1$, $\alpha \in \mathbf{N}$, $V \nmid_f x$. Thus

$$(80) \qquad SxV^{\alpha+t}\omega T = xV^{\alpha+2t}\omega, \quad S \neq 1.$$

Thus $Sx = xy$ and $\omega T = z\omega$ for some $y, z \in \mathcal{F}^1$. Since $S \neq 1$ we get $y \neq 1$. By (80) we get

$$yV^{\alpha+t}z = V^{\alpha+2t}.$$

By Lemma 1.2(iv), $y \in \langle\langle V \rangle\rangle^1$. Since $y \neq 1$ we get $V \mid_f y \mid_f xy = Sx$. Since $V \nmid_f x$ we get $x \mid_f V$. Hence

$$uv\omega = xV^{\alpha+t}\omega \mid_f V^{\alpha+t+1}\omega \mid_f V^{\alpha+2t}\omega \mid_f xV^{\alpha+2t}\omega = uv^2\omega.$$

This proves the lemma.

Lemma 5.3. *Let $\omega_1, \omega_2 \in \mathcal{F}$ such that $|\omega_1| \leq |\omega_2|$, $\omega_1 \nmid \omega_2$ and $\omega_2 \to \omega_1$. Then there exist $a, b, c \in \mathcal{F}^1$ such that $\omega_1 = abc$ and $\omega_2 = bcab$.*

Proof. If $\omega_2 \mid \omega_1$ then since $|\omega_1| \leq |\omega_2|$ we get $\omega_1 = \omega_2$ which implies $\omega_1 \mid \omega_2$, a contradiction. Hence $\omega_2 \nmid \omega_1$. By Lemma 1.3(i) there exist $x, y \in \mathcal{F}^1$, $k \in \mathbf{N}$ such that $x \mid_f \omega_1$, $y \mid_i \omega_1$ and $\omega_2 = x\omega_1^k y$. Since $\omega_1 \nmid \omega_2$ we get $k = 0$. Hence $\omega_2 = xy$. Since $x \mid_f \omega_1$ and $y \mid_i \omega_1$ we get $\omega_1 = ax = yc$ for some $a, c \in \mathcal{F}^1$. Now $|ax| = |\omega_1| \leq |\omega_2| = |xy|$. Hence $|a| \leq |y|$. Thus $y = ab$ for some $b \in \mathcal{F}^1$. This implies that $\omega_1 = abc$ and $x = bc$. Hence $\omega_2 = xy = bcab$. This proves the lemma.

Definition. (1) The class $\mathcal{S} = \{(a_1, a_2, a_3, a_4) | a_1, a_2, a_3, a_4 \in \mathcal{F}(\Omega)^1$ for some non-empty set Ω such that $1 \neq a_1a_2a_3a_4 \mid_i a_4a_1a_2a_2a_3\}$. If $\sigma = (a_1, a_2, a_3, a_4) \in \mathcal{S}$ then $f(\sigma) = (a_1a_2a_3a_4, a_2a_3a_4a_1a_2) \in \mathfrak{T}$.

(2) The class $\mathcal{S}^* = \{(b_4, b_3, b_2, b_1) | b_1, b_2, b_3, b_4 \in \mathcal{F}(\Omega)^1$ for some non-empty set Ω such that $1 \neq b_4b_3b_2b_1 \mid_f b_3b_2b_2b_1b_4\}$. If $\sigma = (b_4, b_3, b_2, b_1) \in \mathcal{S}^*$ then $g(\sigma) = (b_4b_3b_2b_1, b_2b_1b_4b_3b_2) \in \mathfrak{T}$.

Theorem 5.4. *Let $(\omega_1, \omega_2) \in \mathfrak{T}$ such that $(\omega_1, \omega_2) \notin \mathfrak{T}_0$. Then either there exists $\sigma \in \mathcal{S}$ such that $(\omega_1, \omega_2) = f(\sigma)$ or there exists $\sigma \in \mathcal{S}^*$ such that $(\omega_1, \omega_2) = g(\sigma)$.*

Proof. Let $\omega_1, \omega_2 \in \mathcal{F}$. Then $|\omega_1| \leqslant |\omega_2|$, $\omega_2 — \omega_1$, $\omega_1 \nmid \omega_2$. By Lemma 5.3 there exist $a,b,c \in \mathcal{F}^1$ such that $abc = \omega_1$ and $bcab = \omega_2$. Now as $|\omega_1| \leqslant |\omega_2|$ and $\omega_1 \neq \omega_2$ we get $\omega_2 \nmid \omega_1$. By Lemma 1.3(v), $\omega_1 \mid \omega_2^2$. So $u\omega_1 v = \omega_2^2$ for some $u, v \in \mathcal{F}^1$. Thus

$$uabcv = bcabbcab \,. \tag{81}$$

If $|uabc| \leqslant |bcab|$ then $\omega_1 \mid uabc \mid bcab = \omega_2$ a contradiction. So $|uabc| > |bcab|$ which implies $|u| > |b|$. Similarly $|v| > |b|$. Thus $u = bu_1$, $v = v_1 b$ for some $u_1, v_1 \in \mathcal{F}^1$. Hence by (81) we get

$$u_1 abcv_1 = cabbca \,. \tag{82}$$

First let us assume $|u_1| \leqslant |c|$. Then $c = u_1 c_1$ for some $c_1 \in \mathcal{F}^1$. Thus $\omega_1 = abu_1c_1$ and $\omega_2 = bu_1c_1ab$. Also by (82) we get

$$abu_1c_1v_1 = c_1abbu_1c_1a \,.$$

Since $|abu_1c_1| \leqslant |c_1abbu_1|$, we get $abu_1c_1 \mid_{\mathrm{i}} c_1abbu_1$. It follows that $(a, b, u_1, c_1) \in \mathcal{S}$ and $(\omega_1, \omega_2) = f(a,b,u_1,c_1)$. Similarly $|v_1| \leqslant |a|$ implies that $(\omega_1, \omega_2) = g(\sigma)$ for some $\sigma \in \mathcal{S}^*$. So the only case left is when in (82), $|u_1| > |c|$ and $|v_1| > |a|$. Then $u_1 = cx$, $v_1 = ya$ for some $x, y \in \mathcal{F}$. Hence by (82),

$$xabcy = abbc.$$

So $abc \mid ab^2c$ and by Lemma 5.2, $abc \mid_{\mathrm{i}} ab^2c$ or $abc \mid_{\mathrm{f}} ab^2c$. Now $abc \mid_{\mathrm{i}} ab^2c$ implies $(a,b,c,1) \in \mathcal{S}$ and $f(a,b,c,1) = (\omega_1, \omega_2)$. Also $abc \mid_{\mathrm{f}} ab^2c$ implies $(1,a,b,c) \in \mathcal{S}^*$ and $g(1,a,b,c) = (\omega_1, \omega_2)$. This proves the theorem.

The reason for our interest in $\mathcal{S}$ and $\mathcal{S}^*$ is now clear. Of course, studying $\mathcal{S}^*$ is dual to studying $\mathcal{S}$. Consider the following equation in free monoids $\mathcal{F}^1$

$$x_1x_2x_3x_4x_5 = x_4x_1x_2x_2x_3 \,. \tag{83}$$

In (83) x_5 is completely determined by x_1, x_2, x_3 and x_4. So (83) is equivalent to studying

$$x_1x_2x_3x_4 \mid_{\mathrm{i}} x_4x_1x_2x_2x_3 \,. \tag{84}$$

One (trivial) solution is $x_1 = x_2 = x_3 = x_4 = 1$. Thus $\mathcal{S}$ is the class of all non-trivial solutions of (84) in free monoids.

Theorem 5.5. *Let $\omega_1, \omega_2 \in \mathcal{F}$. Then*

(i) *$(\omega_1, \omega_2) = f(a_1, a_2, a_3, 1)$ for some $(a_1, a_2, a_3, 1) \in \mathcal{S}$ if and only if $\omega_1 \equiv \omega_2$ or there exist $a,b,c \in \mathcal{F}^1$, $k \in \mathbf{Z}^+$ such that $\omega_1 = a(bc)^k b$ and $\omega_2 = (bc)^k babc$.*

(ii) *$(\omega_1, \omega_2) = g(1, b_3, b_2, b_1)$ for some $(1, b_3, b_2, b_1) \in \mathcal{S}^*$ if and only if*

$\omega_1 \equiv \omega_2$ or there exist $a, b, c \in \mathcal{F}^1$ such that $\omega_1 = b(cb)^k a$ and $\omega_2 = cbab(cb)^k$.

Proof. (ii) being dual to (i), we need only prove (i). First let $\omega_1 \equiv \omega_2$. Then $\omega_1 = uv$, $\omega_2 = vu$ for some $u, v \in \mathcal{F}^1$. Then $(u, 1, v, 1) \in \mathcal{S}$ and $f(u, 1, v, 1) = (\omega_1, \omega_2)$. Next assume $\omega_1 = a(bc)^k b$ and $\omega_2 = (bc)^k babc$ for some $a, b, c \in \mathcal{F}^1$, $k \in \mathbf{Z}^+$. Then $\sigma = (a, bc, (bc)^{k-1}b, 1) \in \mathcal{S}$ and $f(\sigma) = (\omega_1, \omega_2)$.

Assume conversely that $(\omega_1, \omega_2) = f(a_1, a_2, a_3, 1)$ for some $(a_1, a_2, a_3, 1) \in \mathcal{S}$. Then $\omega_1 = a_1a_2a_3$ and $\omega_2 = a_2a_3a_1a_2$. If $a_2 = 1$ then $\omega_1 \equiv \omega_2$ and we are done. So assume $a_2 \neq 1$. Now $a_1a_2a_3 \,|_{\mathrm{i}}\, a_1a_2a_2a_3$ implying $a_3 \,|_{\mathrm{i}}\, a_2a_3$. Since $a_2 \neq 1$, Lemma 1.1 (vi) implies that there exist $b, c \in \mathcal{F}^1$, $j \in \mathbf{N}$ such that $a_2 = bc$, $a_3 = (bc)^j b$. Then $\omega_1 = a(bc)^{j+1}b$ and $\omega_2 = (bc)^{j+1}babc$ where $a = a_1$. This proves the theorem.

Suppose $\omega_1, \omega_2 \in \mathcal{F}$ and some letter ϵ appears exactly *once* in both ω_1 and ω_2. Then $\omega_1 = a\epsilon v_1$, $\omega_2 = b\epsilon v_2$ for some $a, b, v_1, v_2 \in \mathcal{F}^1$. Set $u_1 = a\epsilon$, $u_2 = b\epsilon$. Then $\omega_1 = u_1v_1$ and $\omega_2 = u_2v_2$. It is then easy to see that $\omega_1 — \omega_2$ if and only if

$$(85) \qquad u_1 \,|_{\mathrm{f}}\, \omega_2u_2, \quad u_2 \,|_{\mathrm{f}}\, \omega_1u_1, \quad v_1 \,|_{\mathrm{i}}\, v_2\omega_2, \quad v_2 \,|_{\mathrm{i}}\, v_1\omega_1.$$

Now let $\omega_1, \omega_2 \in \mathcal{F}$, $u_1, u_2, v_1, v_2 \in \mathcal{F}^1$ such that $\omega_1 = u_1v_1$, $\omega_2 = u_2v_2$. Then it is easy to see that if (85) holds then $\omega_1 — \omega_2$. One might wonder if $\omega_1 — \omega_2$, $|\omega_1| \leqslant |\omega_2|$, $\omega_1 \nmid \omega_2$ implies the existence of u_1, v_1, u_2, v_2 such that (85) holds and $\omega_1 = u_1v_1$, $\omega_2 = u_2v_2$. However, the archimedean relation is too complex for this to be true. On the other hand, this special case of the archimedean relation is characterized by the situations arising in Theorem 5.5.

Theorem 5.6. *Let $\omega_1, \omega_2 \in \mathcal{F}$, $|\omega_1| \leqslant |\omega_2|$, $\omega_1 \nmid \omega_2$. Then there exist $u_1, u_2, v_1, v_2 \in \mathcal{F}^1$ such that $\omega_1 = u_1v_1$, $\omega_2 = u_2v_2$, $u_1 \,|_{\mathrm{f}}\, \omega_2u_2$, $u_2 \,|_{\mathrm{f}}\, \omega_1u_1$, $v_1 \,|_{\mathrm{i}}\, v_2\omega_2$, $v_2 \,|_{\mathrm{i}}\, v_1\omega_1$ if and only if one of the following is true:*

(i) $\omega_1 \equiv \omega_2$,

or

(ii) *There exist $a, b, c \in \mathcal{F}^1$, $k \in \mathbf{Z}^+$ such that $\omega_1 = a(bc)^k b$ and $\omega_2 = (bc)^k babc$,*

or

(iii) *There exist $a, b, c \in \mathcal{F}^1$, $k \in \mathbf{Z}^+$ such that $\omega_1 = b(cb)^k a$ and $\omega_2 = cbab(cb)^k$.*

Proof. First assume (i) holds. So $\omega_1 = ab$, $\omega_2 = ba$ for some $a, b \in \mathcal{F}^1$. Then set $u_1 = a$, $v_1 = b$, $u_2 = ba$, $v_2 = 1$. Next let (ii) hold. Then we set $u_1 = a$, $v_1 = (bc)^k b$, $u_2 = (bc)^k ba$ and $v_2 = bc$. Finally let (iii) hold. Then we set $u_1 = b(cb)^k$, $v_1 = a$, $u_2 = cb$ and $v_2 = ab(cb)^k$.

Assume conversely that $\omega_1 = u_1v_1$, $\omega_2 = u_2v_2$ with u_1, u_2, v_1, v_2 satisfying the prescribed conditions. Now it cannot be that $|u_1| > |u_2|$ and $|v_1| > |v_2|$ because then $|\omega_1| > |\omega_2|$, a contradiction. So

$$|u_1| \leqslant |u_2| \quad \text{or} \quad |v_1| \leqslant |v_2|.$$

By the dual nature of the theorem we may assume $|u_1| \leqslant |u_2|$. Then $u_1 \mid_f u_2$. Now if $|v_1| \leqslant |v_2|$, then $v_1 \mid_i v_2$ which implies $\omega_1 = u_1v_1 \mid u_2v_2 = \omega_2$, a contradiction. Hence $|v_1| > |v_2|$ and $v_2 \mid_i v_1$. Hence

(86) $\quad u_2 = yu_1, \quad v_1 = v_2c \quad$ for some $\quad y, c \in \mathcal{F}^1$.

Now $yu_1 = u_2 \mid_f \omega_1u_1$. Hence

(87) $\quad y \mid_f \omega_1 = u_1v_1 = u_1v_2c$.

Also,

$$|yu_1v_2| = |u_2v_2| = |\omega_2| \geqslant |\omega_1| = |v_1u_1| = |v_2cu_1| .$$

Hence $|y| \geqslant |c|$. By (87) we get $c \mid_f y$. So $y = bc$ for some $b \in \mathcal{F}^1$. So by (87), $bc \mid_f u_1v_2c$ implying $b \mid_f u_1v_2$. Thus $u_1v_2 = ab$ for some $a \in \mathcal{F}^1$. Hence by (86),

(88) $\quad \omega_1 = u_1v_1 = u_1v_2c = abc; \quad \omega_2 = u_2v_2 = yu_1v_2 = bcab$.

Next $v_2c = v_1 \mid_i v_2\omega_2 = v_2bcab$. Hence $c \mid_i bcab$ implying $c \mid_i bc$. Thus $abc \mid_i abbc$. It follows by (88) that $(a,b,c,1) \in \mathcal{S}$ and $f(a,b,c,1) = (\omega_1, \omega_2)$. We are now done by Theorem 5.5.

Let us now get back to studying $\mathcal{S}$ in general. Let $\sigma = (a_1,a_2,a_3,a_4) \in \mathcal{S}$. Then Ω^σ is the (finite) set of letters appearing in $a_1a_2a_3a_4$. Similarly if $\theta = (\omega_1, \omega_2) \in \mathfrak{T}$ then Ω^θ is the (finite) set of letters appearing in $\omega_1\omega_2$. Now let $\sigma_1 = (a_1,a_2,a_3,a_4)$, $\sigma_2 = (b_1,b_2,b_3,b_4) \in \mathcal{S}$. We define $\sigma_1 \equiv \sigma_2$ if there exists a bijection $\varphi: \Omega^{\sigma_1} \to \Omega^{\sigma_2}$ such that the corresponding homomorphism $\hat{\varphi}: \mathcal{F}(\Omega^{\sigma_1})^1 \to \mathcal{F}(\Omega^{\sigma_2})^1$ satisfies $\hat{\varphi}(a_i) = b_i$ $(i = 1,2,3,4)$. The bijection φ, if it exists, is easily seen to be unique. $\equiv$ is evidently an equivalence relation on $\mathcal{S}$. Thus we are thinking of two solutions of (84) to be the same if they are identical except for the renaming of the letters. Similarly if $\theta_1 = (u_1,u_2)$, $\theta_2 = (v_1,v_2) \in \mathfrak{T}$ then we define $\theta_1 \equiv \theta_2$ if there exists a (necessarily) unique bijection $\varphi: \Omega^{\theta_1} \to \Omega^{\theta_2}$ such that the corresponding homomorphism $\hat{\varphi}: \mathcal{F}(\Omega^{\theta_1})^1 \to \mathcal{F}(\Omega^{\theta_2})^1$ satisfies $\hat{\varphi}(u_i) = v_i$ $(i = 1,2)$. $\equiv$ is an equivalence relation on $\mathfrak{T}$. Choosing Ω^σ to always be a finite subset of a fixed countably infinite set we can think of both $\mathcal{S}' = \mathcal{S}/\equiv$ and $\mathfrak{T}' = \mathfrak{T}/\equiv$ as sets.

Let us call a homomorphism $\varphi: \mathcal{F}(\Omega_1)^1 \to \mathcal{F}(\Omega_2)^1$ *proper* if $1 \notin \varphi(\Omega_1)$. We now look at variations of concepts in [3,16] restricted to our special situation. Let $\sigma_1 = (a_1,a_2,a_3,a_4)$, $\sigma_2 = (b_1,b_2,b_3,b_4) \in \mathcal{S}$. We define $\sigma_1 \preccurlyeq \sigma_2$ (σ_2 *follows from* σ_1) if there exists a homomorphism $\varphi: \mathcal{F}(\Omega^{\sigma_1})^1 \to \mathcal{F}(\Omega^{\sigma_2})^1$ such that $\varphi(a_i) = b_i$ $(i = 1,2,3,4)$. We define $\sigma_1 \leqslant \sigma_2$ if φ above is proper. Thus $\leqslant \subseteq \preccurlyeq$. $\leqslant$ and $\preccurlyeq$ are both reflexive and transitive. Let $\sigma_1, \sigma_2 \in \mathcal{S}$. Then $\sigma_1 \leqslant \sigma_2 \leqslant \sigma_1$ if and only if $\sigma_1 \equiv \sigma_2$. Thus $\leqslant$ is a partial order on $\mathcal{S}'$. However $\preccurlyeq$ remains only a preorder even in $\mathcal{S}'$. Let $\sigma \in \mathcal{S}'$. Then σ is said to be *principal* if $\sigma_1 \in \mathcal{S}'$, $\sigma_1 \leqslant \sigma$ implies $\sigma_1 = \sigma$. Also $\sigma \in \mathcal{S}'$ is *strictly principal* if σ is principal and for any $\sigma_1 \in \mathcal{S}'$, $\sigma_1 \preccurlyeq \sigma$ implies $\sigma \preccurlyeq \sigma_1$. Similarly the concepts of 'follows from' ($\preccurlyeq$), 'properly follows from' ($\leqslant$), being principal and strictly principal are defined in $\mathfrak{T}$ and $\mathfrak{T}'$. For instance if $\theta_1 = (u_1, u_2)$,

$\theta_2 = (v_1, v_2) \in \mathfrak{I}$ then we define $\theta_1 \leqslant \theta_2$ (θ_2 properly follows from θ_1) if there exists a proper homomorphism φ: $\mathcal{F}(\Omega^{\theta_1})^1 \to \mathcal{F}(\Omega^{\theta_2})^1$ such that $\varphi(u_i) = v_i$ ($i = 1, 2$).

In what follows, 'Lentin's theorem' refers to [3; Theorem 1.3.16]. There is some confusion in [3] regarding the use of $\leqslant$ and $\preccurlyeq$ so that Lentin's theorem is true with respect to $\leqslant$ but false with respect to $\preccurlyeq$. The situation is more or less remedied in [16]. Also, Professor Lentin has informed the author that the proof of this theorem given in [3] is incomplete but that it is easily completed. According to [16] every $\sigma \in \mathcal{S}'$ follows from some (not necessarily unique) strictly principal $\sigma_1 \in \mathcal{S}'$. More non-trivially, Lentin's theorem implies that every $\sigma \in \mathcal{S}'$ properly follows from a *unique* principal $\sigma_1 \in \mathcal{S}'$. Lentin's theorem can be used to show that if $\sigma = (a_1, a_2, a_3, a_4) \in \mathcal{S}'$ is principal then the greatest common divisor of $|a_1|$, $|a_2|$, $|a_3|$ and $|a_4|$ is 1 (the converse is false). Let $\sigma \in \mathcal{S}'$ be principal. Then since σ is the solution of the equation (83) in five variables, [3] implies that $|\Omega^\sigma| \leqslant 4$. We can improve this bound to 3 and it is the best possible. Also if $\sigma \in \mathcal{S}'$ is principal and $|\Omega^\sigma| = 3$, then σ is strictly principal. Also [3; Chapter 3] gives rise to an algorithm for computing the principal solution associated with any $\sigma \in \mathcal{S}'$. For the convenience of the reader we illustrate this algorithm with an example. Let $\sigma = (a, b, c, d) \in \mathcal{S}'$ where $\Omega^\sigma = \{A, B\}$ and $a = AB^3$, $b = (AB)^2$, $c = B^2(AB)^3$, $d = AB^3AB$. Then

$$abcd \mid_{\mathrm{i}} dabbc$$

$$d = ad_1 \quad \text{where} \quad d_1 = AB$$

$$bcad \mid_{\mathrm{i}} d_1abbc$$

$$b = d_1b_1 \quad \text{where} \quad b_1 = AB$$

$$b_1cad_1 \mid_{\mathrm{i}} ad_1b_1d_1b_1c$$

$$a = b_1a_1 \quad \text{where} \quad a_1 = B^2$$

$$cb_1a_1d_1 \mid_{\mathrm{i}} a_1d_1b_1d_1b_1c$$

$$c = a_1d_1b_1d_1$$

$$b_1a_1d_1 \mid_{\mathrm{i}} b_1a_1d_1b_1d_1 \,.$$

We see that $a = b_1a_1$, $b = d_1b_1$, $c = a_1d_1b_1d_1$ and $d = b_1a_1d_1$. Introducing the letters A_1, A_2, A_3 corresponding to b_1, a_1, d_1 respectively we let $\hat{\sigma} = (u_1, u_2, u_3, u_4)$ where $u_1 = A_1A_2$, $u_2 = A_3A_1$, $u_3 = A_2A_3A_1A_3$ and $u_4 = A_1A_2A_3$. Then $\hat{\sigma} \in \mathcal{S}'$, σ properly follows from $\hat{\sigma}$ (by sending A_1 to b_1, A_2 to a_1 and A_3 to d_1) and $\hat{\sigma}$ is principal.

Let φ: $\mathcal{F}(\Omega_1)^1 \to \mathcal{F}(\Omega_2)^1$ be a *proper* homomorphism and $a, b, c \in \mathcal{F}(\Omega_1)^1$ such that $a \mid_{\mathrm{i}} bc$. Then $a \mid_{\mathrm{i}} b$ if and only if $\varphi(a) \mid_{\mathrm{i}} \varphi(b)$; $b \mid_{\mathrm{i}} a$ if and only if $\varphi(b) \mid_{\mathrm{i}} \varphi(a)$; $a = b$ if and only if $\varphi(a) = \varphi(b)$. This can be used to give the main idea behind Lentin's theorem restricted to our case. The process of [3] that we discussed above associates with each $\sigma \in \mathcal{S}'$ some $\hat{\sigma}$ in $\mathcal{S}'$ such that $\hat{\sigma} \leqslant \sigma$. If $\sigma_1, \sigma_2 \in \mathcal{S}'$ and $\sigma_1 \leqslant \sigma_2$ then the preceding remark can be used to show that $\hat{\sigma}_1 = \hat{\sigma}_2$. Hence $\hat{\sigma}$ is principal for all $\sigma \in \mathcal{S}'$; $\sigma = \hat{\sigma}$ if and only if σ is principal.

Let $\theta \in \mathfrak{T}'$. Then we can show that θ follows from some strictly principal $\theta_1 \in \mathfrak{T}'$ and properly follows from some principal $\theta_2 \in \mathfrak{T}'$. However now neither θ_1 nor θ_2 need be unique. For instance in $\mathcal{F}(A,B,C)$, let $\theta_1 = (ABABA^2B, ABA^2BABA)$, $\theta_2 = (AB, ABA)$, $\theta_3 = (ABCABCACABC, CABCACABCABCA)$, $\theta_4 = (ABC, CABCA)$. Then θ_1, θ_2 are principal and θ_3, θ_4 are strictly principal. Also $\theta_i \leqslant (A^7, A^8)$, $i = 1,2$; $\theta_j \not\leqslant (A^7, A^8)$, $j = 3,4$. If $\theta \in \mathfrak{T}'$ is principal then $|\Omega^\theta| \leqslant 3$. If $\theta \in \mathfrak{T}'$ is principal and $|\Omega^\theta| = 3$ then θ is strictly principal.

Definition. Let $\sigma = (a_1, a_2, a_3, a_4) \in \mathcal{S}'$.

(1) $\eta_1(\sigma) = (a_4a_1, a_2, a_3, a_4)$.

(2) $\eta_2(\sigma) = (a_1, a_2, a_3, a_1a_2a_3a_4)$.

(3) $\eta_3(\sigma) = (a_1, a_2a_3a_4a_1a_2, a_3, a_1a_2a_3a_4)$.

(4) Let $i \in \mathbf{Z}^+$, $\eta_4 = \eta_4^{(i)}$ where $\eta_4(\sigma) = ((a_1a_2^2a_3)^i a_1, a_2, a_3a_4a_1a_2^2a_3, (a_1a_2^2a_3)^i a_1a_2a_3a_4)$.

Remark 5.7. Let $\sigma \in \mathcal{S}'$ and $t \in \{1,2,3,4\}$. Then $\eta_t(\sigma) \in \mathcal{S}'$. Also, σ is principal if and only if $\eta_t(\sigma)$ is.

Example 5.8. We now look at some examples of elements in $\mathcal{S}'$. Let $\mathcal{F} = \mathcal{F}(A,B,C)$.

(1) $\sigma_1 = (AB, A, BA, AB)$.

(2) $\sigma_2 = (A, B, AB^2, AB)$.

(3) $\sigma_3 = ((AB)^2BA, BAB, AB^2, (AB)^2BAB)$.

(4) $\sigma_4 = (AB, CA, BCAC, ABC)$.

(5) Let $i,j,k,l \in \mathbf{N}$, $\sigma_5 = (x_1, x_2, x_3, x_4)$ where

$x_1 = 1$,
$x_2 = (AB)^{i+j}A$,
$x_3 = (BA)^k(AB)^j[(AB)^{i+k}A(AB)^j]^l$,
$x_4 = (AB)^k$.

(6) Let $i,j,k,l,m,n \in \mathbf{N}$ such that $i - l = n - j = k - m$ and $k \leqslant i$. We let $\sigma_6 = (x_1, x_2, x_3, x_4)$ where

$x_1 = C(ABC)^l(BCA)^m$,
$x_2 = (BCA)^iBC$,
$x_3 = (ABC)^j(BCA)^kB$,
$x_4 = C(ABC)^l(BCA)^{m+n}B$.

σ_1, σ_2 and σ_3 are principal. σ_4 is strictly principal. σ_1 is not strictly principal since $\sigma_4 \leqslant \sigma_1$ but $\sigma_1 \not\leqslant \sigma_4$. We do not know exactly when σ_5 is principal but certainly it is necessary that the greatest common divisor of $2i+1$, j and k is 1. Also we suspect that σ_6 is principal whenever j and m are relatively prime. Starting with these various principal solutions in $\mathcal{S}'$ we can construct much more complicated newer ones through the application of the various η_t's.

Remark 5.9. Let $\sigma \in \mathcal{S}'$. Then $f(\sigma)$ is principal implies σ is principal. However, the converse is not true. For instance in $\mathcal{F}(A,B)$ let $\sigma = (1, ABA, BA, AB)$. Then σ is principal. However, $f(\sigma)$ is not principal since $(AB, ABA) < f(\sigma)$. Note that in this case $f\eta_1(\sigma)$ is principal.

Let $\sigma \in \mathcal{S}'$ and $f(\sigma) = (\omega_1, \omega_2)$. Suppose $\omega_1, \omega_2 \in \mathcal{F}$, ω_1 is primitive and $|\{(x,y) \mid x, y \in \mathcal{F}^1, x\omega_1 y = \omega_2^2\}| = 1$. Then σ is principal implies $f(\sigma)$ is principal. Thus we verify that in Example 5.8, $f(\sigma_j)$ is principal for $j = 1,2,3,4$.

Ideally we would like to find an "explicitly" given (in the sense of Lemma 1.1 (iii)) subset $\mathcal{M}$ of $\mathcal{S}'$ so that every $\sigma \in \mathcal{S}'$ follows from some solution in $\mathcal{M}$. This, however, seems unlikely in light of Remark 5.7 and Example 5.8.

6. Free product of positive rationals

Let Ω be a non-empty set. If $A \in \Omega$, $\alpha \in \mathbf{Q}^+$ write A^α for (A, α). Let $\mathcal{F}_\mathbf{Q} = \mathcal{F}_\mathbf{Q}(\Omega)$ be the set of all non-empty finite sequences (called words) of the type $\omega = A_1^{\alpha_1} \dots A_n^{\alpha_n}$ where $n \in \mathbf{Z}^+$, $\alpha_1, \dots, \alpha_n \in \mathbf{Q}^+$, $A_1, \dots, A_n \in \Omega$ and $A_i \neq A_{i+1}$ for $i, i+1 \in \{1, \dots, n\}$. We define $e(\omega) = n$ and $|\omega| = \alpha_1 + \dots + \alpha_n$. Let $\omega_1, \omega_2 \in \mathcal{F}_\mathbf{Q}$. Suppose $\omega_1 = A_1^{\alpha_1} \dots A_n^{\alpha_n}$, $\omega_2 = B_1^{\beta_1} \dots B_m^{\beta_m}$. Then we define

$$\omega_1\omega_2 = \begin{cases} A_1^{\alpha_1} \dots A_n^{\alpha_n + \beta_1} B_2^{\beta_2} \dots B_m^{\beta_m}, & \text{if } A_n = B_1 \\ A_1^{\alpha_1} \dots A_n^{\alpha_n} B_1^{\beta_1} \dots B_m^{\beta_m}, & \text{if } A_n \neq B_1. \end{cases}$$

$\mathcal{F}_\mathbf{Q}(\Omega)$ is a semigroup and is just the free product of $|\Omega|$ disjoint copies of $\mathbf{Q}^+$ under addition (see for example [4; p. 411]). Now, of course, expressions of the type $\omega = A_1^{\alpha_1} \dots A_n^{\alpha_n}$ ($\alpha_1, \dots, \alpha_n \in \mathbf{Q}^+$; $A_1, \dots, A_n \in \Omega$) make sense even when $A_i = A_{i+1}$ for some $i, i+1 \in \{1, \dots, n\}$. But note that if $n = e(\omega)$ then $A_i \neq A_{i+1}$ for any $i, i+1 \in \{1, \dots, n\}$. Now $1 \notin \mathcal{F}_\mathbf{Q}$ and we let $\omega^0 = 1$ for $\omega \in \mathcal{F}_\mathbf{Q}^1$. Also we let $|1| = e(1) = 0$.

Let $Y = \{A_1, \dots, A_m\}$ be a finite non-empty set. Then $\mathcal{C}_\mathbf{Q} = \mathcal{C}_\mathbf{Q}(Y)$ is the subsemigroup of $\mathcal{F}_\mathbf{Q}(Y)$ consisting of all words involving each A_i at least once ($i = 1, \dots, m$). $\mathcal{C}_\mathbf{Q}(Y)$ is an ideal of $\mathcal{F}_\mathbf{Q}(Y)$. If $\omega_1, \omega_2 \in \mathcal{C}_\mathbf{Q}(Y)$, then $\omega_1 — \omega_2$ in $\mathcal{C}_\mathbf{Q}(Y)$ if and only if $\omega_1 — \omega_2$ in $\mathcal{F}_\mathbf{Q}(Y)$. When we talk about $|$, $|_\mathrm{i}$, $|_\mathrm{f}$ for elements in $\mathcal{C}_\mathbf{Q}(Y)$ we are talking about these relations in $\mathcal{F}_\mathbf{Q}(Y)$ (equivalently $\mathcal{F}_\mathbf{Q}(\Omega)^1$ where $Y \subseteq \Omega$). Let Ω be a non-empty set. It follows by [8, 11] that the connected components of ($\mathcal{F}_\mathbf{Q}(\Omega)$,—) are exactly $\mathcal{C}_\mathbf{Q}(Y)$ as Y ranges through all finite non-empty subsets of Ω. Let us also note that $\mathcal{F}(\Omega)$ is a subsemigroup of $\mathcal{F}_\mathbf{Q}(\Omega)$, and if Ω is finite then $\mathcal{C}(\Omega)$ is a subsemigroup of $\mathcal{C}_\mathbf{Q}(\Omega)$.

While most of Section 1 generalizes to $\mathcal{F}_\mathbf{Q}$, we only state what we need.

Lemma 6.1. (i) *Let $u_1, u_2, v_1, v_2 \in \mathcal{F}_\mathbf{Q}^1$ such that $u_1u_2 = v_1v_2$. Then $|u_1| \leqslant |v_1|$ if and only if $|v_2| \leqslant |u_2|$ if and only if $u_1 \,|_\mathrm{i}\, v_1$ if and only if $v_2 \,|_\mathrm{f}\, u_2$. $|u_1| = |v_1|$ if and only if $|u_2| = |v_2|$ if and only if $u_1 = v_1$ if and only if $u_2 = v_2$.*

(ii) *Let* $\omega_1, \omega_2 \in \mathcal{F}_Q$ *such that* $\omega_1 \to \omega_2$. *Then there exist* $j \in \mathbf{Z}^+$, $u, v \in \mathcal{F}_Q^1$ *such that* $|u|, |v| < |\omega_2|$ *and* $\omega_2^j = u\omega_1 v$. *If further* $\omega_1 \nmid \omega_2$ *then there exist* $x, y \in \mathcal{F}^1$, $k \in \mathbf{N}$ *such that* $|x|, |y| < |\omega_2|$, $x \mid_f \omega_2$, $y \mid_i \omega_2$ *and* $\omega_1 = x\omega_2^k y$.

Definition. Let $\omega_1, \omega_2 \in \mathcal{F}_Q$.

(1) $\omega_1 \equiv \omega_2$ if $\omega_1 = ST$, $\omega_2 = TS$ for some $S, T \in \mathcal{F}_Q^1$.

(2) $\omega_1 \sim \omega_2$ if $\omega_1 = (ST)^i$, $\omega_2 = (TS)^j$ for some $S, T \in \mathcal{F}_Q^1$ and $i, j \in \mathbf{Z}^+$.

Remark 6.2. Let Ω be a non-empty set. Then

(1) Let $\rho \in \{|, |_i, |_f, —, \equiv, \sim\}$ and let $\omega_1, \omega_2 \in \mathcal{F}_Q(\Omega)$. Then for any automorphism ψ of $\mathcal{F}_Q$, ψ extends uniquely to an automorphism of $\mathcal{F}_Q^1$; $\omega_1 \rho \omega_2$ if and only if $\psi(\omega_1) \rho \psi(\omega_2)$.

(2) Let $t \in \mathbf{Z}^+$, $A_1, ..., A_t \in \Omega$, $\epsilon_1, ..., \epsilon_t \in \mathbf{Q}^+$. If the A_i's are distinct then there exists an automorphism φ of $\mathcal{F}_Q(\Omega)$ such that $\varphi(A_i) = A_i^{\epsilon_i}$ $(i = 1, ..., t)$ and $\varphi(B) = B$ for all other $B \in \Omega$. If Y is any finite subset of Ω then φ restricted to $\mathcal{C}_Q(Y)$ is an automorphism of $\mathcal{C}_Q(Y)$.

(3) Let $\mathcal{A}$ be a finite subset of $\mathcal{F}_Q(\Omega)$. Then there exists an automorphism φ of $\mathcal{F}_Q(\Omega)$ of the type described in (2) such that $\varphi(\mathcal{A}) \subseteq \mathcal{F}(\Omega)$.

Lemma 6.3. (i) *Let* $\rho \in \{|, |_i, |_f, \to, —\}$. *Then for* $\omega_1, \omega_2 \in \mathcal{F}$, $\omega_1 \rho \omega_2$ *in* $\mathcal{F}$ *if and only if* $\omega_1 \rho \omega_2$ *in* $\mathcal{F}_Q$.

(ii) *Let* $\omega_1, \omega_2 \in \mathcal{F}_Q$. *Then* $\omega_1 \equiv \omega_2$ *if and only if* $|\omega_1| = |\omega_2|$ *and* $\omega_1 \to \omega_2$.

(iii) *Let* $\omega_1, \omega_2, \omega_3 \in \mathcal{F}_Q$ *such that* $\omega_1 \to \omega_2 \sim \omega_3$. *Then* $\omega_1 \to \omega_3$.

(iv) *Let* $\omega_1, \omega_2 \in \mathcal{F}_Q$. *Then* $\omega_1 \sim \omega_2$ *if and only if* $\omega_1^i \to \omega_2$ *for all* $i \in \mathbf{Z}^+$.

(v) *The relations* $\equiv$ *and* $\sim$ *are equivalence relations on* $\mathcal{F}_Q$ *and* $\equiv, \subseteq \sim \subseteq —$.

(vi) *Let* $\omega_1, \omega_2 \in \mathcal{F}$ *and* $\rho \in \{\equiv, \sim\}$. *Then* $\omega_1 \rho \omega_2$ *in* $\mathcal{F}$ *if and only if* $\omega_1 \rho \omega_2$ *in* $\mathcal{F}_Q$.

(vii) *Suppose* $\omega_1, \omega_2 \in \mathcal{F}_Q$, $\omega_1 \sim \omega_2$ *and* $\omega_1 \not\equiv \omega_2$. *Then* $\omega_1 \mid \omega_2$ *or* $\omega_2 \mid \omega_1$.

Proof. (i) For $\rho \in \{|, |_i, |_f\}$ the result follows by induction on $|\omega_2|$. Then the rest follows trivially.

(ii) This follows by Lemma 6.1 (ii) similar to Lemma 1.4 (i).

(iii) Trivial.

(iv) Suppose $\omega_1^i \to \omega_2$ for all $i \in \mathbf{Z}^+$. There exists an automorphism φ of $\mathcal{F}_Q$ such that $\varphi(\omega_1)$, $\varphi(\omega_2) \in \mathcal{F}$. Also for all $i \in \mathbf{Z}^+$, $\varphi(\omega_1)^i \to \varphi(\omega_2)$ in $\mathcal{F}_Q$ and hence in $\mathcal{F}$. By Lemma 1.4(v), $\varphi(\omega_1) \sim \varphi(\omega_2)$ in $\mathcal{F}$ and hence in $\mathcal{F}_Q$. So $\omega_1 \sim \omega_2$. The converse is trivial.

(v) Follows from (ii), (iii) and (iv).

(vi) Follows from (i), (ii), (iv) and Lemma 1.4.

(vii) Proof is similar to Lemma 1.4 (viii).

Definition. For $\alpha \in \mathbf{Q}$ we let α^+, α^- denote the smallest integer $\geqslant \alpha$ and the greatest integer $\leqslant \alpha$, respectively. Let $\omega \in \mathcal{F}_Q(\Omega)$ and $n = e(\omega)$. Let $\omega = A_1^{\alpha_1} ... A_n^{\alpha_n}$ where

$\alpha_1, ..., \alpha_n \in \mathbf{Q}^+$ and $A_1, ..., A_n \in \Omega$. Then $\omega^+ = A_1^{\alpha_1^+} ... A_n^{\alpha_n^+}$ and $\omega^- = A_1^{\alpha_1^-} ... A_n^{\alpha_n^-}$. So $\omega^+ \in \mathcal{F}$, $\omega^- \in \mathcal{F}^1$.

If $\omega \in \mathcal{F}_\mathbf{Q}$ it can happen that $\omega^- = 1$. If $\omega \in \mathcal{C}_\mathbf{Q}$ then $\omega^+ \in \mathcal{C}_\mathbf{Q}$. If $\omega \in \mathcal{F}$ then $\omega^+ = \omega^- = \omega$.

Lemma 6.4. *Let $\omega_1, \omega_2 \in \mathcal{F}_\mathbf{Q}$ such that $\omega_1 \mid \omega_2$. Then $\omega_1^+ \mid \omega_2^+$ and $\omega_1^- \mid \omega_2^-$.*

Proof. Let $\mathcal{F}_\mathbf{Q} = \mathcal{F}_\mathbf{Q}(\Omega)$, $\omega_1 = A_1^{\alpha_1} ... A_n^{\alpha_n}$ where $n = e(\omega_1)$. Let $B \in \Omega$, $\beta \in \mathbf{Q}^+$. Set $\omega = \omega_1 B^\beta$. If $B \neq A_n$ then $\omega^+ = \omega_1^+ B^{\beta^+}$ and $\omega^- = \omega_1^- B^{\beta^-}$. If $B = A_n$ then $\omega^+ = A_1^{\alpha_1^+} ... A_n^{(\alpha_n+\beta)^+}$ and $\omega^- = A_1^{\alpha_1^-} ... A_n^{(\alpha_n+\beta)^-}$. Since $\alpha_n^+ \leqslant (\alpha_n + \beta)^+$ and $\alpha_n^- \leqslant (\alpha_n + \beta)^-$ we see that in any case $\omega_1^+ \mid_\mathrm{i} \omega^+$ and $\omega_1^- \mid_\mathrm{i} \omega^-$. Now let $v \in \mathcal{F}_\mathbf{Q}$, $e(v) = m$, $v = B_1^{\beta_1} ... B_m^{\beta_m}$. Set $v_i = B_1^{\beta_1} ... B_i^{\beta_i}$ $(i = 1, ..., m)$. Then by the above,

$$\omega_1^+ \mid_\mathrm{i} (\omega_1 v_1)^+ \mid_\mathrm{i} (\omega_1 v_2)^+ \mid_\mathrm{i} \cdots \mid_\mathrm{i} (\omega_1 v_m)^+ = (\omega_1 v)^+;$$

$$\omega_1^- \mid_\mathrm{i} (\omega_1 v_1)^- \mid_\mathrm{i} \cdots \mid_\mathrm{i} (\omega_1 v_m)^- = (\omega_1 v)^-.$$

Thus $\omega_1^+ \mid_\mathrm{i} (\omega_1 v)^+$ and $\omega_1^- \mid_\mathrm{i} (\omega_1 v)^-$ for all $v \in \mathcal{F}_\mathbf{Q}^1$. Similarly $\omega_1^+ \mid_\mathrm{f} (u\omega_1)^+$ and $\omega_1^- \mid_\mathrm{f} (u\omega_1)^-$ for all $u \in \mathcal{F}_\mathbf{Q}^1$. Hence $\omega_1^+ \mid_\mathrm{i} (\omega_1 v)^+ \mid_\mathrm{f} (u\omega_1 v)^+$ and $\omega_1^- \mid_\mathrm{i} (\omega_1 v)^- \mid_\mathrm{f} (u\omega_1 v)^-$ for all $u, v \in \mathcal{F}_\mathbf{Q}^1$. So $\omega_1^+ \mid (u\omega_1 v)^+$ and $\omega_1^- \mid (u\omega_1 v)^-$ for all $u, v \in \mathcal{F}_\mathbf{Q}^1$. This proves the lemma.

Lemma 6.5. *There exists a function $\tau: \mathcal{F}_\mathbf{Q} \to \mathcal{F}$ such that τ is identity on $\mathcal{F}$ and for all $\omega \in \mathcal{F}_\mathbf{Q}$, $\tau(\omega) \mid \omega^+$ and for all $i \in \mathbf{Z}^+$, $(\omega^i)^+ \to \tau(\omega)$. So for all $\omega_1, \omega_2 \in \mathcal{F}_\mathbf{Q}$, $\omega_1 — \omega_2$ implies $\tau(\omega_1) — \tau(\omega_2)$.*

Proof. Let $\mathcal{F}_\mathbf{Q} = \mathcal{F}_\mathbf{Q}(\Omega)$ and let $\omega \in \mathcal{F}_\mathbf{Q}$, $e(\omega) = n$. Then there exist $A_1, ..., A_n \in \Omega$, $\alpha_1, ..., \alpha_n \in \mathbf{Q}^+$ such that $\omega = A_1^{\alpha_1} ... A_n^{\alpha_n}$. If $n = 1$ we can let $\tau(\omega) = \omega^+$. So let $n > 1$. If $A_n \neq A_1$ then $(\omega^i)^+ = (\omega^+)^i$ for all $i \in \mathbf{Z}^+$ and we can let $\tau(\omega) = \omega^+$. So let $A_n = A_1$. Now $(\alpha_1 + \alpha_n)^+ = \alpha_1^+ + \alpha_n^+$ or $\alpha_1^+ + \alpha_n^+ - 1$. Let $\beta_n = \alpha_n^+$ if $(\alpha_1 + \alpha_n)^+ = \alpha_1^+ + \alpha_n^+$ and let $\beta_n = \alpha_n^+ - 1$ if $(\alpha_1 + \alpha_n)^+ = \alpha_1^+ + \alpha_n^+ - 1$. So $\beta_n \in \mathbf{N}$ and $\alpha_1^+ + \beta_n = (\alpha_1 + \alpha_n)^+$. We let $\tau(\omega) = A_1^{\alpha_1^+} ... A_{n-1}^{\alpha_{n-1}^+} A_n^{\beta_n}$. So $\tau(\omega) \mid \omega^+$ and for all $i \in \mathbf{Z}^+$, $(\omega^i)^+ \mid \tau(\omega)^{i+1}$. Next let $\omega_1, \omega_2 \in \mathcal{F}_\mathbf{Q}$ such that $\omega_1 — \omega_2$. So $\omega_1 \mid \omega_2^i$ for some $i \in \mathbf{Z}^+$. Hence $\tau(\omega_1) \mid \omega_1^+ \mid (\omega_2^i)^+ \to \tau(\omega_2)$. So $\tau(\omega_1) \to \tau(\omega_2)$. Similarly $\tau(\omega_2) \to \tau(\omega_1)$ and $\tau(\omega_1) — \tau(\omega_2)$.

Theorem 6.6. (i) *Suppose $\omega_1, \omega_2 \in \mathcal{F}$ and $\langle U_i \rangle_{i=1}^i$ a minimal sequence between ω_1 and ω_2 in $\mathcal{F}$. Then $\langle U_i \rangle_{i=1}^n$ is minimal in $\mathcal{F}_\mathbf{Q}$.*

(ii) *Let $\omega_1, \omega_2 \in \mathcal{C}_\mathbf{Q}$, $\omega_1 \not— \omega_2$. Let φ be an automorphism of $\mathcal{C}_\mathbf{Q}$ such that $\varphi(\omega_1), \varphi(\omega_2) \in \mathcal{C}$. Then $\varphi(\omega_1) \not— \varphi(\omega_2)$. If $\langle U_i \rangle_{i=1}^n$ is any minimal sequence between $\varphi(\omega_1)$ and $\varphi(\omega_2)$ in $\mathcal{C}$ then $\langle \varphi^{-1}(U_i) \rangle_{i=1}^n$ is a minimal sequence between ω_1 and ω_2 in $\mathcal{C}_\mathbf{Q}$.*

Proof. (i) Let $\langle V_j\rangle_{j=1}^m$ be a sequence between ω_1 and ω_2 in $\mathcal{F}_\mathbf{Q}$. By Lemma 6.5, $\langle\tau(V_j)\rangle_{j=1}^m$ is a sequence between $\tau(\omega_1)$ and $\tau(\omega_2)$ in $\mathcal{F}$. Since $\omega_1, \omega_2 \in \mathcal{F}$ we get $\omega_1 = \tau(\omega_1)$ and $\tau(\omega_2) = \omega_2$. By minimality of $\langle U_i\rangle_{i=1}^n$ we get $n \leqslant m$. So $\langle U_i\rangle_{i=1}^n$ is minimal in $\mathcal{F}_\mathbf{Q}$.

(ii) By (i) $\langle U_i\rangle_{i=1}^n$ is minimal in $\mathcal{C}_\mathbf{Q}$. So $\langle\varphi^{-1}(U_i)\rangle_{i=1}^n$ is minimal in $\mathcal{C}_\mathbf{Q}$.

Remark 6.7. Let $\omega_1, \omega_2 \in \mathcal{C}$, $\omega_1 \not\!\!-\!\!- \omega_2$. Then [9; Theorem 3.3, Theorem 4.4] tells us (at least theoretically) how to find all minimal sequences between ω_1 and ω_2 in $\mathcal{C}$. Thus if $\omega_1, \omega_2 \in \mathcal{C}_\mathbf{Q}$, $\omega_1 \not\!\!-\!\!- \omega_2$ then Remark 6.2 and Theorem 6.6 give us an algorithm for finding at least one minimal sequence between ω_1 and ω_2 in $\mathcal{C}_\mathbf{Q}$. The problem of finding an algorithm for obtaining a description of all minimal sequences between ω_1 and ω_2 in $\mathcal{C}_\mathbf{Q}$ remains open.

Lemma 6.8. *Let $\omega_1, \omega_2 \in \mathcal{C} = \mathcal{C}(\Omega)$, $\omega_1 \equiv \omega_2$, $\omega_1 \neq \omega_2$. Then there exists $V \in \mathcal{C}_\mathbf{Q}$ such that $\omega_1 \mid V$, $\omega_1 \text{---} V$, $\omega_2 \not\rightarrow V$ and so that for any $\omega \in \mathcal{C}$, $\omega \rightarrow V$ implies $\omega \rightarrow \omega_1$, and $\omega \text{---} V$ implies $\omega \text{---} \omega_1$.*

Proof. $\omega_1 = W_1^i$, $\omega_2 = W_2^j$ for some $i, j \in \mathbf{Z}^+$, some primitive $W_1, W_2 \in \mathcal{C}$. Since $\omega_1 \equiv \omega_2$, we get $W_1 \sim W_2$ and by Lemma 1.4, $W_1 \equiv W_2$. Since $\omega_1 \equiv \omega_2$ we get $i = j$ and $\omega_1 = W_1^j$, $\omega_2 = W_2^j$. Since $\omega_1 \neq \omega_2$ we get $W_1 \neq W_2$. So since $|W_1| = |W_2|$ we get $e(W_1) \geqslant 2$. Now $W_1 = ST$, $W_2 = TS$ for some $S, T \in \mathcal{F}^1$. Since $W_1 \neq W_2$ we get $S, T \in \mathcal{F}$. There are three possible cases: (i) $e(T) \geqslant 2$, or (ii) $e(S) \geqslant 2$, or (iii) S and T both consist of single but different letters. (ii) being dual to (i) we may assume that (i) or (iii) holds. In either case, if S starts with the letter A, then some letter other than A appears in T. So $S = AL$, $T = MBA^\alpha$ for some $A, B \in \Omega$, $A \neq B$, $L, M \in \mathcal{F}^1$, $\alpha \in \mathbf{N}$. So

$$\omega_1 = (ALMBA^\alpha)^j; \quad \omega_2 = (MBA^{\alpha+1}L)^j.$$

Let $\epsilon = \frac{1}{2}$ and $V = B^\epsilon A^\alpha \omega_1 A^\epsilon$. Then $\omega_1 \mid V$ and $\omega_1 \text{---} V$.

Now let $\omega \in \mathcal{F}$ such that $\omega \rightarrow V$. Let $V_1 = A^\epsilon B^\epsilon A^\alpha \omega_1 \equiv V$. So $\omega \rightarrow V_1$. We claim that $\omega \mid V_1 A^\epsilon B^\epsilon$. So suppose not. Then $\omega \nmid V_1$. By Lemma 6.1 (ii) there exist $x, y \in \mathcal{F}^1$, $k \in \mathbf{N}$ such that $\omega = x V_1^k y$ and $x \mid_\mathrm{f} V_1$, $y \mid_\mathrm{i} V_1$. Since $\omega \in \mathcal{F}$ we get $A^\epsilon B^\epsilon A \nmid \omega$. Since $A \mid_\mathrm{i} \omega_1$ we obtain that $A^\epsilon B^\epsilon A \mid_\mathrm{i} A^\epsilon B^\epsilon A^\alpha \omega_1 = V_1$ which implies $V_1 \nmid \omega$. So $k = 0$ and $\omega = xy$. If $|y| > |A^\epsilon B^\epsilon|$ then since $y \mid_\mathrm{i} V_1$ and $A^\epsilon B^\epsilon A \mid_\mathrm{i} V_1$ we obtain that $A^\epsilon B^\epsilon A^\delta \mid_\mathrm{i} y$ for some $\delta \in \mathbf{Q}^+$. So $A^\epsilon B^\epsilon A^\delta \mid \omega$ contradicting the fact that $\omega \in \mathcal{F}$. Thus $|y| \leqslant |A^\epsilon B^\epsilon|$ and $y \mid_\mathrm{i} A^\epsilon B^\epsilon$. Since $x \mid_\mathrm{f} V_1$ we get $\omega = xy \mid V_1 A^\epsilon B^\epsilon$. Thus for $\omega \in \mathcal{F}$, $\omega \rightarrow V$ implies $\omega \mid V_1 A^\epsilon B^\epsilon$. But then by Lemma 6.4,

$$\omega = \omega^- \mid (V_1 A^\epsilon B^\epsilon)^- = (A^\epsilon B^\epsilon A^\alpha \omega_1 A^\epsilon B^\epsilon)^- = A^\alpha \omega_1.$$

So we have (since $A^\alpha \omega_1 \rightarrow \omega_1$):

(89) For $\omega \in \mathcal{F}$, $\omega \rightarrow V$ implies that $\omega \mid A^\alpha \omega_1$ and $\omega \rightarrow \omega_1$.

Now let $\omega \in \mathcal{G}$, ω — V. Then since $\omega_1 \mid V$ we get $\omega_1 \to \omega$. Also by (89) $\omega \to \omega_1$. Hence ω — ω_1. Finally we claim that $\omega_2 \nrightarrow V$. For suppose $\omega_2 \to V$. Then by (89), $\omega_2 \mid A^\alpha\omega_1$. So there exist $u, v \in \mathcal{F}^1$ such that $u\omega_2 v = A^\alpha \omega_1$. Hence

$$u(MBA^{\alpha+1}L)^j v = A^\alpha(ALMBA^\alpha)^j.$$

Thus

$$(MBu)\, W_2^j(vAL) = MBA^\alpha(ALMBA^\alpha)^j AL = W_2^{j+1}.$$

Since W_2 is primitive and $MBu, vAL \neq 1$ we get by Lemma 1.2(iv) that $MBu, vAL \in \langle\langle W_2 \rangle\rangle$. So $|MBu|, |vAL| \geqslant |W_2|$. This contradicts the above. Hence $\omega_2 \nrightarrow V$ and the lemma is proved.

Definition. Let $\mathcal{L}$ be a finite non-empty subset of $\mathcal{F}_\mathbf{Q}$.

(1) $\mathcal{L}$ is a line if $\mathcal{L}$ consists of distinct points $u_1, \ldots, u_t$ $(t \in \mathbf{Z}^+)$ such that if $1 \leqslant i < j \leqslant t$, then u_i — u_j if and only if $j = i + 1$. If further $u_1 \nsim u_i$ for $i \neq 1$ and $u_i \nsim u_t$ for $t \neq i$ then $\mathcal{L}$ is a *-line.

(2) $\mathcal{L}$ is a union of lines if $\mathcal{L}$ is a disjoint union of non-empty subsets $\mathcal{L}_j$ $(j = 1, \ldots, m)$ such that each $\mathcal{L}_j$ is a line and $a \in \mathcal{L}_j$, $b \in \mathcal{L}_k$, $j \neq k$ implies $a \nmid\!\!- b$. If further each $\mathcal{L}_j$ is a *-line then we say $\mathcal{L}$ is a union of *-lines.

(3) By a polygon $\mathcal{P}$ in $\mathcal{F}_\mathbf{Q}$ we mean a set of n distinct points $u_1, \ldots, u_n \in \mathcal{F}_\mathbf{Q}$, $n \geqslant 3$ such that u_1 — u_2 — ... — u_n — u_1. $\mathcal{P}$ is irreducible if $1 \leqslant i < j \leqslant n$ and u_i — u_j imply $j = i + 1$, or $i = 1$ and $j = n$. By a polygon through $\mathcal{L}$ we mean a polygon whose set of vertices contains $\mathcal{L}$.

Lemma 6.9. *Let $\mathcal{L} \subseteq \mathcal{C}$ be a line such that for all $u, v \in \mathcal{L}$, $u \sim v$ implies $u \equiv v$. Then $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{C}_\mathbf{Q}$ where $\mathcal{L}'$ is a *-line such that for any $\omega \in \mathcal{C}$, $u \nmid\!\!- \omega$ for all $u \in \mathcal{L}$ implies $u \nmid\!\!- \omega$ for all $u \in \mathcal{L}'$.*

Proof. If $|\mathcal{L}| = 1$ then $\mathcal{L}$ is a *-line. So let $|\mathcal{L}| = n \geqslant 2$. Then $\mathcal{L} = \{u_1$ — u_2 — ... — $u_n\}$. Assume $\mathcal{L}$ is not a *-line. By symmetry let $u_1 \sim u_\alpha$ for some $\alpha > 1$. So u_1 — u_α and $\alpha = 2$. Hence $u_1 \sim u_2$. By hypothesis $u_1 \equiv u_2$. By Lemma 6.8 there exists $V_1 \in \mathcal{C}_\mathbf{Q}$ such that

(90) $\quad u_1 \mid V_1,\ u_1$ — $V_1,\ u_2 \nrightarrow V_1$; for any $\omega \in \mathcal{C}$, $\omega \to V_1$ implies $\omega \to u_1$, and ω — V_1 implies ω — u_1.

It follows that $\mathcal{L}_1 = \{V_1$ — u_1 — u_2 — ... — $u_n\}$ is a line. If $V_1 \sim u_\beta$ for some β then $\beta = 1$ and $V_1 \sim u_1$. Since $u_1 \equiv u_2$ we get $V_1 \sim u_2$ implying V_1 — u_2, a contradiction. Now if $u_n \nsim u_\gamma$ for $\gamma < n$ then $\mathcal{L}_1$ is a *-line and we are done. So assume $u_\gamma \sim u_n$ for some $\gamma < n$. Then u_γ — u_n. Hence $\gamma = n - 1$ and $u_{n-1} \sim u_n$. By hypothesis $u_{n-1} \equiv u_n$. By Lemma 6.8 there exists $V_2 \in \mathcal{C}_\mathbf{Q}$ such that (90) holds with V_1 replaced by V_2, u_1 replaced by u_n and u_2 replaced by u_{n-1}. We claim $V_2 \nmid\!\!- V_1$. For suppose V_2 — V_1. Then since $u_n \mid V_2$ we get $u_n \to V_1$. By (90), $u_n \to u_1$. Similarly $u_1 \to u_n$ and u_1 — u_n. But then $n = 2$ and by above $u_2 \to V_1$

contradicting (90). So $V_2 \not\!\!— V_1$. It follows that $\mathcal{L}' = \{V_1 — u_1 — ... — u_n — V_2\}$ is a line. If $u_\delta \sim V_2$ for some δ then $u_\delta — V_2$ and $\delta = n$. Hence $u_n \sim V_2$. Since $u_{n-1} \equiv u_n$ we get $u_{n-1} \sim V_2$ implying $u_{n-1} — V_2$, a contradiction. Hence $\mathcal{L}'$ is a $*$-line. By (90) and the dual property for V_2, we see that the lemma is proved.

Lemma 6.10. *Let $\mathcal{L} \subseteq \mathcal{C}_\mathbf{Q}$ be a union of lines such that for $u, v \in \mathcal{L}$, $u \sim v$ implies $u \equiv v$. Then $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{C}_\mathbf{Q}$ such that $\mathcal{L}'$ is a $*$-line.*

Proof. First we show that $\mathcal{L}$ is contained in a union of $*$-lines. Let $\mathcal{L}$ be disjoint union of lines $\mathcal{L}_i$ $(i = 1, ..., m)$ such that $a \in \mathcal{L}_i$, $b \in \mathcal{L}_j$, $i \neq j$ implies $a \not\!\!— b$. By Remark 6.2 there exists an automorphism φ of $\mathcal{C}_\mathbf{Q}$ such that $\mathcal{M} = \varphi(\mathcal{L}) \subseteq \mathcal{C}$. Let $\varphi(\mathcal{L}_i) = \mathcal{M}_i$. By Lemma 6.9 there exists a $*$-line $\mathcal{M}_1'$ such that $\mathcal{M}_1 \subseteq \mathcal{M}_1'$ and $\omega \in \mathcal{C}$, $\omega — u' \in \mathcal{M}_1'$ implies $\omega — u$ for some $u \in \mathcal{M}_1$. Hence $\omega \not\!\!— v$ for any $v \in \mathcal{M}_1'$, $\omega \in \mathcal{M}_i$, $i > 1$. Let $\mathcal{L}_1' = \varphi^{-1}(\mathcal{M}_1')$. So $\mathcal{L}_1 \subseteq \mathcal{L}_1'$ is a $*$-line and $v \in \mathcal{L}_1'$, $\omega \in \mathcal{L}_i$, $i > 1$, implies $\omega \not\!\!— v$. Now by hypothesis for any $u, v \in \mathcal{L}_2$, $u \sim v$ implies $u \equiv v$. So we can repeat the process for $\mathcal{L}_2$ to obtain a $*$-line $\mathcal{L}_2'$. Continuing we obtain a union of $*$-lines $\mathcal{N} = \mathcal{L}_1' \cup ... \cup \mathcal{L}_n'$ such that $\mathcal{L} \subseteq \mathcal{N}$. There exists an automorphism ψ of $\mathcal{C}_\mathbf{Q}$ such that $\mathcal{N}_1 = \psi(\mathcal{N}) \subseteq \mathcal{C}$. Evidently $\mathcal{N}_1$ is a union of $*$-lines. By Lemma 2.5, $\mathcal{N}_1 \subseteq \mathcal{N}' \subseteq \mathcal{C}$ such that $\mathcal{N}'$ is a $*$-line. So $\mathcal{L} \subseteq \psi^{-1}(\mathcal{N}') = \mathcal{L}' \subseteq \mathcal{C}_\mathbf{Q}$ and $\mathcal{L}'$ is a $*$-line. This proves the lemma.

Lemma 6.11. *Let $\mathcal{P}$ be an irreducible n-gon in $\mathcal{F}_\mathbf{Q}$, $n \geqslant 4$. Suppose $u, v \in \mathcal{P}$ and $u \not\equiv v$. Then $u \not\sim v$.*

Proof. Suppose $u \sim v$. Then by Lemma 6.3 (vii), $u \mid v$ or $v \mid u$. By symmetry let $u \mid v$. Since $u \sim v$ we get $u — v$. Then since $n \geqslant 4$ there must exist $\omega \in \mathcal{P}$ such that $v — \omega$, $u \not\!\!— \omega$. But $u \to \omega$ (since $u \mid v$) and $\omega \to u$ (since $v \sim u$). This contradiction proves the lemma.

Theorem 6.12. *Let $\mathcal{L}$ be a finite non-empty subset of $\mathcal{C}_\mathbf{Q} = \mathcal{C}_\mathbf{Q}(A_1, ..., A_t)$, $t \geqslant 2$. Then the following are equivalent.*

(i) *$\mathcal{L}$ is a union of lines and for any $u, v \in \mathcal{L}$, $u \sim v$ implies $u \equiv v$.*

(ii) *There exists $k \in \mathbf{Z}^+$ such that for any $l \in \mathbf{Z}^+$, $l \geqslant k$ there exists an irreducible l-gon in $\mathcal{C}_\mathbf{Q}$ through $\mathcal{L}$.*

(iii) *There exists an irreducible n-gon through $\mathcal{L}$ for some $n \in \mathbf{Z}^+$ such that $n \geqslant 4$ and $n \geqslant |\mathcal{L}| + 1$.*

Proof. (i) $\Rightarrow$ (ii). By Lemma 6.10, $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{C}_\mathbf{Q}$ such that $\mathcal{L}'$ is a $*$-line. There exists an automorphism φ of $\mathcal{C}_\mathbf{Q}$ such that $\mathcal{M} = \varphi(\mathcal{L}') \subseteq \mathcal{C}$. Evidently $\mathcal{M}$ is a $*$-line. By the proof of Theorem 2.6 there exists $k \in \mathbf{Z}^+$ such that for each $l \geqslant k$ there exists an irreducible l-gon $\mathcal{P}$ in $\mathcal{C}$ through $\mathcal{M}$. But then $\varphi^{-1}(\mathcal{P})$ is an irreducible l-gon through $\mathcal{L}$ in $\mathcal{C}_\mathbf{Q}$.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) ⇒ (i). Since $\mathcal{L}$ is a proper subset of an irreducible polygon, it must be a union of lines. The rest follows from Lemma 6.11.

Remark 6.13. Let $\mathcal{F}_{\mathbf{R}}$, $\mathcal{C}_{\mathbf{R}}$ be the semigroups obtained by replacing $\mathbf{Q}^+$ by $\mathbf{R}^+$ in the definitions of $\mathcal{F}_{\mathbf{Q}}$ and $\mathcal{C}_{\mathbf{Q}}$. Then Theorem 6.12 remains true for $\mathcal{C}_{\mathbf{R}}(A_1, ..., A_t)$ where $t \geq 2$. But the proof is more difficult. Let $\omega_1, \omega_2 \in \mathcal{C}_{\mathbf{R}}$, $\omega_1 \not\!\!-\!\!- \omega_2$. Does there exist an algorithm which explicitly gives at least one minimal sequence between ω_1 and ω_2 (Theorem 6.6 does this for $\mathcal{C}_{\mathbf{Q}}$).

7. More on induced subgraphs

In this section we show that any finite graph obtained by putting together trees, polygons and complete graphs in certain trivial ways is contained in the free semigroup. We first notice that Remark 6.2 implies the following.

Theorem 7.1. *The finite induced subgraphs of $\mathcal{F}$ are, within isomorphism, the same as the finite induced subgraphs of $\mathcal{F}_{\mathbf{Q}}$.*

Remark 7.2. We do not know whether Theorem 7.1 is true if $\mathbf{Q}$ is replaced by $\mathbf{R}$. However, the counter-example of Section 4 can be shown to be non isomorphic to any induced subgraph of $\mathcal{F}_{\mathbf{R}}(\Omega)$ for any non-empty set Ω.

Theorem 7.1. is false for infinite graphs as the following example shows.

Example 7.3. In $\mathcal{F}_{\mathbf{Q}}(A,B)$ let $\omega_1 = A^2B$, $\omega_2 = A^3B$. For $i \in \mathbf{Z}^+$ let $u_i = A^2BA^{1+1/i}$. Then $\omega_1 \not\!\!-\!\!- \omega_2$ and $\omega_1 — u_i — u_j — \omega_2$ for all $i, j \in \mathbf{Z}^+$. However, Example 3.5 shows that the induced subgraph, $\{\omega_1, \omega_2\} \cup \{u_i \mid i \in \mathbf{Z}^+\}$ is not contained in the free semigroup.

Definition. (1) Let $\mathcal{A} \subseteq \mathcal{F}_{\mathbf{Q}}$. Then $\mathcal{A}$ is discrete if $\omega_1, \omega_2 \in \mathcal{A}$, $\omega_1 \sim \omega_2$ implies $\omega_1 = \omega_2$.

(2) Let Γ be a finite graph. Then Γ is strongly contained in the free semigroup if Γ is connected and for any non-empty set Ω, any $\omega \in \mathcal{F}(\Omega)$ with $e(\omega) \geq 2$ and any $\alpha \in \Gamma$ there exists an injection $\varphi: \Gamma \to \mathcal{F}_{\mathbf{Q}}(\Omega)$ such that

(i) For all $\alpha_1, \alpha_2 \in \Gamma$, $\alpha_1 — \alpha_2$ if and only if $\varphi(\alpha_1) — \varphi(\alpha_2)$,

(ii) $\varphi(\alpha) = \omega$,

(iii) $\omega \mid \varphi(\beta)$ for all $\beta \in \Gamma$, and

(iv) $\varphi(\Gamma)$ is discrete.

Lemma 7.4. *Let $\omega_1, \omega_2 \in \mathcal{F}(\Omega)$, $B \in \Omega$. Suppose B appears in ω_1 and ω_2 and it appears the same number of times. Then $\omega_1 \sim \omega_2$ implies $\omega_1 \equiv \omega_2$.*

Proof. $\omega_1 = (ST)^i$, $\omega_2 = (TS)^j$ for some $i, j \in \mathbf{Z}^+$ and $S, T \in \mathcal{F}^1$. Let B appear k times in ST. Then B appears ki times in ω_1 and kj times in ω_2. So $ki = kj > 0$. Hence $k > 0$ and $i = j$. Thus $\omega_1 \equiv \omega_2$.

Theorem 7.5. *Let $n \in \mathbf{Z}^+$. Then the complete graph with n elements is strongly contained in the free semigroup.*

Proof. Let $\omega \in \mathcal{F}_\mathbf{Q}(\Omega)$, $e(\omega) \geqslant 2$. By Remark 6.2 there exists an automorphism φ of $\mathcal{F}_\mathbf{Q}$ such that $u = \varphi(\omega) \in \mathcal{F}(\Omega)$ and $e(u) \geqslant 2$. Let u start with $A \in \Omega$. Let $u_1 = u$ and $u_i = uA^{1/i}$ $(i = 2, ..., n)$. Then $u \mid u_i$ $(i = 1, ..., n)$ and u_i—u_j for $i, j \in \{1, ..., n\}$. Let ψ be an automorphism of $\mathcal{F}_\mathbf{Q}$ given by $\psi(A) = A^{n!}$ and $\psi(C) = C$ for $C \in \Omega$, $C \neq A$. So $v_i = \psi(u_i) \in \mathcal{F}$ and $|v_i| \neq |v_j|$ for $i, j \in \{1, ..., n\}$, $i \neq j$. Hence $v_i \not\equiv v_j$ for $i, j \in \{1, ..., n\}$, $i \neq j$. There exists $B \in \Omega$, $B \neq A$ such that B appears in u. Then B occurs in v_1 and B occurs the same number of times in each v_i $(i = 1, ..., n)$. By Lemma 7.4, $v_i \not\to v_j$ (hence $u_i \not\to u_j$) for $i, j \in \{1, ..., n\}$, $i \neq j$. So $\{u_i \mid i = 1, ..., n\}$ is discrete. Hence if $a_i = \varphi^{-1}(u_i)$ then $\{a_i \mid i = 1, ..., n\}$ is discrete, $\omega = a_1 \mid a_i$ for all $i \in \{1, ..., n\}$ and a_i—a_j for all $i, j \in \{1, ..., n\}$. By the symmetric nature of points in a complete graph, we are done.

Theorem 7.6. *Every polygon is strongly contained in the free semigroup.*

Proof. Let $n \in \mathbf{Z}^+$, $n \geqslant 3$. We are to show an n-gon is strongly contained in the free semigroup. If $n = 3$ we are done by Theorem 7.5. So let $n \geqslant 4$. Let $\omega \in \mathcal{F}_\mathbf{Q}(\Omega)$ with $e(\omega) \geqslant 2$. Because of the symmetry of the points in an n-gon, we just have to produce a discrete, irreducible n-gon in $\mathcal{F}_\mathbf{Q}$ through ω such that every word in the n-gon contains ω as a segment.

By Remark 6.2 there exists an automorphism φ of $\mathcal{F}_\mathbf{Q}$ such that $u = \varphi(\omega) \in \mathcal{F}(\Omega)$ and $e(u) \geqslant 2$. So $u = V^t$ for some primitive $V \in \mathcal{F}$, $t \in \mathbf{Z}^+$. Since $e(u) \geqslant 2$ we get $|V| > 1$. Let V start with $\epsilon \in \Omega$. Then $|\epsilon| = 1$. Hence $\epsilon \notin \langle\langle V \rangle\rangle^1$. By Lemma 1.2(vii) we get

$$V\epsilon V \not\leftrightarrow V. \tag{91}$$

Let $j \in \mathbf{Z}^+$. We claim that $V^{2^j+4} \not\leftrightarrow V^{2^j}\epsilon$. For suppose $V^{2^j+4} \to V^{2^j}\epsilon$. Then $V\epsilon V \to V^{2^j}\epsilon$ and $|V^{2^j}\epsilon| + |V\epsilon V| \leqslant |V^{2^j+4}|$. By Lemma 1.3(vi), $V\epsilon V \mid V^{2^j+4}$ contradicting (91). So

$$V^{2^j+4} \not\leftrightarrow V^{2^j}\epsilon \quad \text{for all} \quad j \in \mathbf{Z}^+. \tag{92}$$

Since $V^{2^{j+1}}\epsilon \equiv V^{2^j}\epsilon V^{2^j}$ we get by (92)

$$V^{2^{j+1}+4} \not\leftrightarrow V^{2^j}\epsilon V^{2^j}, \quad V^{2^{j+2}} \not\leftrightarrow V^{2^j}\epsilon V^{2^j}, \quad V^{2^{j+2}} \not\leftrightarrow V^{2^{j+1}}\epsilon \quad \text{for all} \quad j \in \mathbf{Z}^+. \tag{93}$$

Now let $r = t + 2 \geqslant 3$. Let $m = n - 4 \in \mathbf{N}$ and for $i = 0, ..., m$ let $v_i = V^{2^{r+i}}\epsilon V^{2^{r+i}}$.

We see by (91), (92) and (93) that the following is an irreducible polygon:

$$V — V^{2^r}\epsilon — v_0 — \dots — v_m — V^{2^{r+m+4}}\epsilon — V.$$

Now $u = V^t$, $t \leqslant r$. Hence $u — V^{2^r}\epsilon$ and $u — V^{2^{r+m+4}}\epsilon$. Also $u_1 \in \mathcal{F}, u — u_1$ implies $V — u_1$. So the following is an irreducible polygon:

$$\mathcal{P} = \{u — V^{2^r}\epsilon — v_0 — \dots — v_m — V^{2^{r+m+4}}\epsilon — u\}.$$

Also, since $r \geqslant 3$ we have that for $u_1, u_2 \in \mathcal{P}$, $u_1 \neq u_2$ implies $|u_1| \neq |u_2|$. Hence $u_1 \nmid u_2$ for $u_1, u_2 \in \mathcal{P}$ with $u_1 \neq u_2$. By Lemma 6.11, $\mathcal{P}$ is discrete. Evidently $|\mathcal{P}| = m + 4 = n$ and $u = V^t \mid u_1$ for all $u_1 \in \mathcal{P}$. Since $\omega = \varphi^{-1}(u)$, $\varphi^{-1}(\mathcal{P})$ is the required irreducible n-gon.

Definition. (1). Let Γ be a graph and Γ_1, Γ_2 induced subgraphs of Γ. Then $\Gamma = \Gamma_1 \circ \Gamma_2(\xi)$ if $\Gamma_1 \cap \Gamma_2 = \{\xi\}$ and for $\alpha_1 \in \Gamma_1$, $\alpha_2 \in \Gamma_2$, $\alpha_1 — \alpha_2$ if and only if $\alpha_1 = \alpha_2 = \xi$.

(2) Let Λ_1, Λ_2 be disjoint graphs, $\alpha \in \Lambda_1$, $\beta \in \Lambda_2$. Then $\Lambda_1 * \Lambda_2(\alpha, \beta) = \Lambda_1 \cup \Lambda_2$. If $\gamma_1, \gamma_2 \in \Lambda_i$ ($i = 1$ or 2) we define $\gamma_1 — \gamma_2$ if and only if $\gamma_1 — \gamma_2$ in Λ_i. If $\gamma_1 \in \Lambda_1$ and $\gamma_2 \in \Lambda_2$ then we define $\gamma_1 — \gamma_2 — \gamma_1$ if and only if $\gamma_1 = \alpha$ and $\gamma_2 = \beta$.

Theorem 7.7. (i) *Let Γ be a finite graph such that $\Gamma = \Gamma_1 \circ \Gamma_2(\xi)$. Suppose that Γ_1 and Γ_2 are strongly contained in the free semigroup. Then Γ is strongly contained in the free semigroup.*

(ii) *Let Λ_1, Λ_2 be disjoint finite graphs strongly contained in the free semigroup such that $\alpha \in \Lambda_1$ and $\beta \in \Lambda_2$. Then $\Lambda = \Lambda_1 * \Lambda_2(\alpha, \beta)$ is strongly contained in the free semigroup.*

Proof. (i) Γ_1, Γ_2 and hence Γ must be connected. Let $\alpha \in \Gamma_1$ (the case $\alpha \in \Gamma_2$ being dual). Let $\omega \in \mathcal{F}_{\mathbf{Q}}$, $e(\omega) \geqslant 2$. Then there exists an injection $\varphi: \Gamma_1 \to \mathcal{F}_{\mathbf{Q}}$ such that for $x, y \in \Gamma_1$, $x — y$ if and only if $\varphi(x) — \varphi(y)$; $\varphi(\alpha) = \omega \mid \varphi(\beta)$ for all $\beta \in \Gamma_1$; $\mathcal{A} = \varphi(\Gamma)$ is discrete. Let $u = \varphi(\xi)$. Since Γ_1 is connected and $e(\omega) \geqslant 2$ we get $e(u) \geqslant 2$. Since $\mathcal{A}$ is discrete, Lemma 6.3 implies that there exists $n \in \mathbf{Z}^+$ such that $u^n \nrightarrow v$ for all $v \in \mathcal{A}$ with $v \neq u$. There exists an injection $\psi : \Gamma_2 \to \mathcal{F}_{\mathbf{Q}}$ such that for $x, y \in \Gamma_2$, $x — y$ if and only if $\psi(x) — \psi(y)$; $\psi(\xi) = u^n \mid \psi(\gamma)$ for all $\gamma \in \Gamma_2$; $\mathcal{B} = \psi(\Gamma_2)$ is discrete. Let $v_1 \in \mathcal{A}$, $v_2 \in \mathcal{B}$, $v_1 \neq u$. Since $u^n \nrightarrow v_1$ and $u^n \mid v_2$ we get $v_2 \nrightarrow v_1$. Hence

(94) $\qquad v_2 \not— v_1 \quad$ for all $\;v_1 \in \mathcal{A},\; v_2 \in \mathcal{B},\; v_1 \neq u.$

Also if $v \in \mathcal{B}$ then since $u^n \mid v$ we see that $u^n — v$ if and only if $u — v$. Let $\mathcal{B}_1 = \mathcal{B} \setminus \{u^n\}$. If $v \in \mathcal{B}_1$ then $v \nrightarrow u^n$ which implies $v \nrightarrow u$. Hence $\mathcal{A} \cup \mathcal{B}_1$ is discrete. Also if $v \in \mathcal{B}_1$ then $\omega | u | u^n | v$. So $\omega | v$ for all $v \in \mathcal{A} \cup \mathcal{B}_1$. By the above discussion, $\mathcal{A} \cup \mathcal{B}_1$ is isomorphic to Γ with the isomorphism θ given by $\theta(\gamma) = \varphi(\gamma)$ if $\gamma \in \Gamma_1$, and $\theta(\gamma) = \psi(\gamma)$ if $\gamma \notin \Gamma_1$.

(ii) Let Γ be the induced subgraph $\{\alpha, \beta\}$ of Λ. By Theorem 7.5, Γ is strongly contained in the free semigroup. Let Λ_3 be the induced subgraph $\Lambda_1 \cup \{\beta\}$. Then by (i), $\Lambda_3 = \Lambda_1 \circ \Gamma(\alpha)$ is strongly contained in the free semigroup. Again by (i), $\Lambda = \Lambda_3 \circ \Lambda_2(\beta)$ is strongly contained in the free semigroup.

Corollary 7.8. *Every finite tree is strongly contained in the free semigroup.*

Remark 7.9. It is not known to us whether every finite planar graph is contained in the free semigroup. Certainly not every connected planar graph is strongly contained in the free semigroup. In fact let $\Gamma = \{0,1,2,3,4\}$ where for $i,j \in \Gamma$ we set $i \text{---} j$ if and only if $i = j$ or $\{i,j\} \in \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,2\},\{2,3\},\{3,4\},\{4,1\}\}$. It can in fact be shown that Γ is not isomorphic to any induced subgraph of $\mathcal{F}_Q(A,B)$ containing AB at the center. On the other hand Γ is contained in the free semigroup. In fact in $\mathcal{F}(A,B)$ let $\omega_0 = ABA^2B$, $\omega_1 = (AB)^2A^2B$, $\omega_2 = (AB)^2A(AB)^2A$, $\omega_3 = ABA(AB)^2A^2B$, $\omega_4 = ABA(AB)^2A$. Then it is readily verified that for $i,j \in \Gamma$, $i \text{---} j$ if and only if $\omega_i \text{---} \omega_j$.

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Note added in the proof. In addition to [3] there is also the book by Ju. I. Hmelevskii, Equations in the free semigroup, Trudy Matem. Inst. im. Steklova 107 (1971) 288 pages (in Russian).